Necessary and Sufficient Conditions for Weak Singularity in Tolman-Bondi Models

[a,b]Bina R. Patel[1], [a]A.H. Hasmani[2]
[a] Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar-388120, Gujarat, India
[b] Department of mathematical sciences, P. D. Patel Institute of Applied Sciences, Charotar University of Science and Technology (CHARUSAT), Changa-388421, Gujarat, India

Abstract:

We examine the weak shell-crossing singularity in the inhomogeneous Tolman-Bondi model in the absence of a cosmological constant. We find that weak shell-crossing singularity occurs during gravitational collapse when the outer shells go faster than inner shells of radius, which are sufficient conditions. Initial data can lead to weak singularity. The pressure of shell-crossing weak singularity determines by initial velocity, by the different method we examine the condition on mass and radius $\frac{F'}{F} > \frac{3}{r}$.

Keywords:

Cosmological models; Inhomogeneous universe; Weak Singularity;

AMS Subject Classification: 83D05, 83F05

1 Introduction

Tolman-Bondi model describes the gravitational collapse of spherically symmetric dust matter distribution. Tolman-Bondi model matched to Schwarzschild exterior where all g^{ij} are functions of C^{∞} type. Initial density and velocity in the Tolman-Bondi model are functions of radial coordinate r only. The Tolman-Bondi model's collapse is pressureless, which means every particular shell of dust with finite radius will collapse through its Schwarzschild radius.

For the homogeneous cloud, all shells of matters are not defined at the same time and, thus there is no weak singularity at all Oppenheimer-Snyder [6]. The proper time for inhomogeneous matter distribution depends on radius (co-moving coordinate) r; as shell-crossings are not genuine curvature singularities, the nearby shell of matter operates developing momentary density singularity, where Kretschmann curvature scalar cloud blows up, this can be removed through the extension of spacetime. In natural objects, density, and pressure, is enormous; this may be the reason for weak singularity. In the Tolman-Bondi model, shell-crossing singularities are not a general singularity, and it is removable. However, detailed analysis by Szekeres and Lum [9] considered that Newtonian and relativistic spherically symmetric matter distribution and they suggested the following notes;

ISSN NO : 0363-8057

(1) Jacobi fields approach the singularity having finite limits.

(2) The boundary region can be transformed by a C^1 transformation.

This allows one to think that such a shell-cross can be possibly avoided if the shapes of the arbitrary functions available in the geometry are properly chosen.

2 TOLMAN-BONDI SPACETIME

As mentioned earlier, the Tolman-Bondi Model represents a spherically symmetric dust matter cloud that is inhomogeneous in the radial direction. Tolman-Bondi model is written in synchronous co-moving coordinates so that $g_{ti} = 0 (i = r, \theta, \phi)$, and $g_{tt} = -1$. For matter particles the velocity vector is $u^i \equiv (1, 0, 0, 0)$, which means that coordinate time and proper time t are same for all particles. The cosmological constant Λ is zero. The spherically symmetric Tolman-Bondi class of solution given by metric below;

$$ds^2 = \quad -dt^2 + e^{-2\nu(t,r)} dr^2 + R^2(t,r) d\Omega^2, \tag{1}$$

where $\nu(t,r)$ and is an arbitrary function and

$$d\Omega^2 = \quad d\theta^2 + \sin^2\theta d\phi^2. \tag{2}$$

The material content of the spacetime is assumed to be dust so that stress-energy tensor,

$$T_{ij} = \rho(t,r)\ u_i u_j, \tag{3}$$

where $\rho(t,r)$ denoting the energy density and only non-vanishing component of energy-momentum tensor is $T_{00} = \rho$. Introducing new auxiliary functions,

$$f(t,r) = \quad e^{2\nu} R'^2(t,r) - 1, \tag{4}$$

$$F(t,r) = \quad R(t,r)(\dot{R}^2 - f). \tag{5}$$

This simplifies Einstein's equations greatly to,

$$\dot{R}^2 = \quad \frac{F}{R(t,r)} + f, \tag{6}$$

$$\dot{f} = \quad 0, \tag{7}$$

$$\dot{F} = \quad 0, \tag{8}$$

with the constraint

$$F' = R^2 R' T_{00}. \tag{9}$$

In view of equations (7) and (8), F and f are functions of r only. The metric (1) with $e^{2\nu} = [1 + f(r)]/R'^2$, together with equations (6) to (9), fully determine the Tolman-Bondi family of solutions.
Thus the Tolman-Bondi metric is,

$$ds^2 = -dt^2 + \frac{R'^2(t,r)}{1+f(r)} dr^2 + R^2(t,r)\ d\Omega^2. \tag{10}$$

Parametric form of Tolman-Bondi family solutions are given below;

Hyperbolic, $f(r) > 0$

$$R(t,r) = \frac{F(r)}{2f(r)}(\cosh\eta - 1), \qquad (\sinh\eta - \eta) = \frac{2f(r)^{3/2}(t-a_0)}{F(r)}; \tag{11}$$

Parabolic, $f(r) = 0$

$$R(t,r) = \left[\frac{9F(r)(t-a_0)^2}{4}\right]^{1/3}; \tag{12}$$

Elliptic, $f(r) < 0$

$$R(t,r) = \frac{F(r)}{-2f(r)}(1-\cos\eta), \qquad (t-a_0) = \frac{F(r)}{2}(-f(r))^{-3/2}(\eta - \sin\eta). \tag{13}$$

As mentioned above Tolman-Bondi model contains three types of evolution with the time that is given by equations (11) to (13). The positive expansion rate ($\dot{R}(r,t) > 0$) this all these models leads to big-bang at $t = a_0(r)$.

The density for Tolman-Bondi metric is given by,

$$8\pi\rho(t,r) = \frac{F'(r)}{R'(t,r)R^2(t,r)}, \tag{14}$$

and the Kretschmann scalar $\mathcal{K} = R^{hijk}R_{hijk}$ is,

$$\mathcal{K} = 12\frac{F^2(r)}{R^6(r,t)} - 8\frac{F(r)F'(r)}{R^5(t,r)R'(t,r)} - 3\frac{F'^2(r)}{R^4(t,r)R'^2(t,r)}, \tag{15}$$

(e.g. Bondi 1947 [16]), a_0, f, and F are arbitrary functions of r, and all these functions have the physical meaning as in the big-bang region the time $t \geq a_0$, in the big crunch region time $t \leq a_0$, and the local time at $R(t,r) = 0$ is $a_0(r)$ and $F(r)$ is the Misner-Sharp mass function that is two times of effective mass m. Since $F(r)$ is related to mass function, it must be non-negative everywhere.

$$F(r) \geq 0. \tag{16}$$

The Conditions for Shell-Crossing Singularity

The shell-crossing are defined by,

$$R' = 0 \quad and \quad R > 0.$$

For general ($f(r) < 0$) Tolman-Bondi solution can be easily obtained by parametric integrations. From equation (13) we can write,

$$(t - a_0) = \frac{F}{2}(-f)^{-3/2}(\eta - \sin\eta), \qquad (17)$$

$$R(t,r) = \frac{F}{f}\sin^2\frac{\eta}{2}. \qquad (18)$$

Where $0 < \eta \leq \pi$ (elliptic), and a_0 is an arbitrary constant of integration that can be fixed with initial data. We can fix $a_0(r)$ with initial data $\dot{R}(0,r) \equiv \phi(r) = a_0(r)$. Therefore equation (17) becomes,

$$a_0 = \frac{F}{2}(-f)^{-3/2}(\eta_0 - \sin\eta_0), \qquad (19)$$

from equation (6) and (18)

$$\frac{F}{R(t,r)} = \dot{R}^2(t,r) - f, \qquad (20)$$

and,

$$R(t,r) = \frac{F}{f}\sin^2\eta/2, \qquad (21)$$

therefore,

$$f^{-3/2} = \frac{(\csc^2\eta/2 + 1)^{-3/2}}{\dot{R}^3} \qquad (22)$$

$$a_0(r) = \frac{F}{2}\nu^{-3}(\csc^2\eta_0/2 + 1)^{-3/2}(\eta_0 - \sin\eta_0), \qquad (23)$$

where η_0 is value of η at $t = 0$ and $\dot{R}^3(0,r) = \nu(r)$. For time symmetric initial data $v(r) = t_0(r) = 0$, which implies $f = \frac{-F}{R}$ as $\dot{R}(0,r) = 0$. The radial coordinate r is merely a different shell, and we can therefore fix the radial coordinate using the initial area radius coordinate radius,

$$R(0,r) = r,$$

so, equations (17) and (18) get simplified by using

$$f = -\frac{F}{r} \quad \text{at} \quad v(r) = a_0(r) = 0, \text{and} \quad R(0,r) = r.$$

$$t(\eta,r) = \left(\frac{r^3}{4F}\right)^{1/2}(\eta - \sin\eta), \qquad (24)$$

$$R(\eta,r) = r\sin^2\frac{\eta}{2}. \qquad (25)$$

A shell with an initial proper area $4\pi r^2$, will thus collapse to vanishing area radius in a time

$$t_{collapse}(r) = \pi\sqrt{\frac{r^3}{4F}}, \tag{26}$$

the relevant derivative of (26) with respect to r is,

$$t'_{collapse}(r) = \frac{\pi}{4}\left(\frac{3}{r} - \frac{F'}{F}\right). \tag{27}$$

When the entire matter collapses to zero radius, the outsides shell of matter goes faster than the insides shell of matter, and at some surface, they intersect each other and create momentary density singularity. The necessary and sufficient condition for shell crossing is the function (26) should be a decreasing function of time. That is,

$$t'_{collapse} < 0, \tag{28}$$

Because of (28), the condition on Misner-sharp mass function holds if and only if

$$\frac{F'}{F} > \frac{3}{r}. \tag{29}$$

This is the shell-crossing condition on mass and physical radius when the collapse occurs, physical radius going to zero, and mass function going to infinity.
For general ($f(r) = 0$) the Tolman-Bondi solution can be easily obtained by parametric integrations. In the Tolman-Bondi model, a marginally bound case deserves special examination. This is the boundary between the hyperbolic region and the elliptic region; also, η is not valid here. However, the parabolic region is a special case with the energy function equal to zero. We can consider this as the most straightforward case too. From equation (6) for parabolic case, we can write,

$$\dot{R}^2(t,r) = \frac{F}{r}, \tag{30}$$

which is integrated to,

$$R(t,r) = \left[\frac{9F(r)(t - a_0(r))^2}{4}\right]^{1/3},$$

here that local time at $R(t,0) = a_0(r)$, as $t \geq a_0(r)$ is a time of big-bang, and in region $t \leq a_0(r)$, this is a time of big-crunch. Without loss of generality we consider a special case $a_0(r) = a_c(r) = \sqrt{\frac{4r^3}{9F}}$. we can write,

$$R(t,r) = \left[\frac{9F(r)(t - a_c(r))^2}{4}\right]^{1/3}, \tag{31}$$

which trivially leads to,

$$R(t,r) = r\left[1 - \frac{t}{a_c(r)}\right]^{2/3}, \tag{32}$$

where,

$$a_c(r) = \sqrt{\frac{4r^3}{9F}}. \tag{33}$$

The time $a_c(r) = \sqrt{\frac{4r^3}{9F}}$ is the proper time for the collapse of a spherical shell with an initial radius r, which is always positive. Here we are concerned only with weak shell-crossing singularity, at the point $R(t,r) > 0$ and $R'(t,r) = 0$. The weak singularity occurs when,

$$R'(t,r) = \left[r \left(1 - \frac{t}{a_c(r)} \right)^{2/3} \right]' \tag{34}$$

$$= \left[1 - \frac{t}{a_c(r)} \right]^{-1/3} \left[1 - \frac{t}{a_c(r)} + a_c(r)\gamma(r) \right] = 0 \tag{35}$$

where $\gamma(r) \equiv (1 - \frac{rF'}{3F})$.
The time when shell-crossing occurs is given by.

$$t = t_s = a_c(1 + a_c\gamma), \tag{36}$$

when the collapse commences, the necessary and sufficient condition for weak singularity is

$$t_s < a_c, \tag{37}$$

This leads to

$$\gamma(r) < 0, \tag{38}$$

i.e.,

$$\frac{F'}{F} > \frac{3}{r}. \tag{39}$$

This is the shell-crossing condition on mass and physical radius in (39), which is the same as the one for the time-symmetric $f(r) < 0$ cases. From equation (32) for any value of $a_c(r)$, this collapse will give a strong shell-focusing singularity at the center.

3 Conclusion

We have shown that excluding the cosmological constant will not prevent shell-crossing singularities from occurring near the center. We have shown that the absence of cosmological constant in the Tolman-Bondi model shell-crossing singularity occurs with several necessary and sufficient conditions. This can explaining by fact that if $t'_{collapse} < 0$ then $\frac{F'}{F} > \frac{3}{r}$ then weak singularity will occurs.

References

[1] A. Meszaros, On Shell-Crossing in the Tolman Metric, Mon. Not. R. Astr. Soc. Vol. 253, No. 619 (1991).

[2] A.H. Hasmani and Bina R. Patel, Simultaneous Weak Singularity and Strong Curvature Singularity in Tolman-Bondi Model with $k(r) = 0$, International Journal of Computer Applications, Vol. 176, No. 7 (2020)

[3] Bina R. Patel and G. M. Deheri, Extension of Some Common Fixed Point Theorems, International Journal of Applied Physics and Mathematics, Vol. 6, No. 172 (2016)

[4] A.H. Hasmani and Bina R. Patel, Shell-Crossing and Shell-Focusing Singularity in Spherically Symmetric Spacetime, PRAJNA - Journal of Pure and Applied Sciences, Vol. 27, No. 70 (2019)

[5] C. Hellaby, Some properties of singularities in the Tolman model, phdthesis, Queen's University at Kingston Ontario, Canada (1985).

[6] C. Hellaby and K. Lake, Shell Crossings and the Tolman Model Astrophys. J. Vol.290 No. 381(1985).

[7] C. Hellaby and K. Lake, Erratum - the Redshift Structure of the Big-Bang in Inhomogeneous Cosmological Models - Part One - Spherical Dust Solutions Astrophys. J. Vol.282 No. 1(1985).

[8] H. Bondi,Spherically symmetrical models in general relativity, Mon. Not. Roy. Astron. Soc., Vol. 107, No. 410 (1947)

[9] J. R Oppenheimer and H. Snyder, On Continued Gravitational Contraction, Phys. Rev. Vol.56, No. 455 (1939)

[10] K. Lake, Precursory singularities in spherical gravitational collapse, Phys. Rev. Lett. Vol.68, No. 3129 (1992)

[11] A.Ori, Inevitability of shell crossing in the gravitational collapse of weakly charged dust spheres, Phys. Rev. D. Vol.44, No.2278 (1991)

[12] P. Szekeres and A. Lun, What is a shell-crossing singularity?, J. Austral. Math. Soc. Ser. B. Vol.41 No.167(1999)

[13] P. S. Joshi and I. H. Dwivedi, Naked singularities in spherically symmetric inhomogeneous Tolman-Bondi dust cloud collapse. Phys. Rev. D Vol. 47 No.5357.(1993)

[14] P. S. Joshi, Gravitational Collapse and Spacetime Singularities Cambridge University Press,Cambridge (2007).

[15] P. S. Joshi, Global Aspects in Gravitational and cosmology, Oxford Uni. press Inc., New York (1993).

[16] P. S. Joshi, Shell-Crossing in gravitational Collapse, Int. Jour. of Mod. Phy., Vol. 22, No. 5 (2013)

[17] R.C.Tolman, Effect of Inhomogeneity on Cosmological Models. Proc. Nat. Acad. Sci., Vol. 20, No. 169 (1934)

[18] S. W. Hawking and G. F. R.Ellis,The Cosmic Black-Body Radiation and the Existence of Singularities in Our Universe. G.F.R, Astrophys. J., Vol. 152, NO. 25 (1968)

[19] H. Bondi, Spherically Symmetrical Models in General Relativity, Monthly Notices of the Royal Astronomical Society, Vol. 107, No. 5 (1947)