Stability of Undecic Functional Equation in Matrix Normed Spaces

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Abstract

In this paper, we define and find the general solution of the following undecic functional equation in matrix normed spaces

$$\begin{aligned} &f(u+6v)-11f(u+5v)+55f(u+4v)-165f(u+3v)+330f(u+2v)\\ &-462f(u+v)+462f(u)-330f(u-v)+165f(u-2v)-55f(u-3v)\\ &+11f(u-4v)-f(u-5v)=11!f(v), \end{aligned}$$

where $11! = 39916800$. Moreover, we also investigate the generalized Ulam-Hyers stability of this undecic functional equation using the fixed point method in matrix normed spaces.

Keywords and phrases: Matrix normed spaces, undecic mapping, Ulam-Hyers stability, fixed point.

1 Introduction

The stability problem of functional equations started from a famous talk of Ulam [20] in 1940 in which he asked a question concerning the stability of homomorphisms. In the next year Hyers [9] gave an affirmative answer for Banach spaces. The result of Hyers was further generalized by Aoki [1] for additive mappings. In 1978, an approach was made to weaken the condition for the Cauchy difference by Rassias [19]. It was further generalized by many mathematicians to investigate stablity of different types of functional equations in various spaces. For detailed information, one can refer ([3–8, 10, 14–18, 21–24]).

Ravi et al. [17] studied the general solution of undecic functional equation and proved the stability of the functional equation in quasi-β-normed spaces by using the fixed point method. From the reference of their this paper we investigate the general solution of undecic functional equation and also proved the stability of this functional equation in matrix normed spaces by using the fixed point method. In this article, we introduce undecic functional equation

$$\begin{aligned} &f(u+6v)-11f(u+5v)+55f(u+4v)-165f(u+3v)+330f(u+2v)\\ &-462f(u+v)+462f(u)-330f(u-v)+165f(u-2v)-55f(u-3v)\\ &+11f(u-4v)-f(u-5v)=11!f(v), \end{aligned} \tag{1.1}$$

where $11! = 39916800$, having solution $f(u) = cu^{11}$ and then find stability in matrix normed spaces.

2 Preliminaries

In this section, we will use the fallowing notations:
$M_n(A)$ is the set of all $n \times n$ matrices in A;
$e_j \in M_{1,n}(\mathbb{C})$ such that the j-th component is 1 and the other components are zero;
$E_{ij} \in M_n(\sigma)$ such that the (i, j)-component is 1 and the other components are zero;
$E_{ij} \otimes u \in M_n(A)$ such that the (i, j)-component is x and the other compments are zero;

For $u \in M_n(A)$, $v \in M_k(A)$

$$u \oplus v = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

Definition 2.1 ([11])**.** Let A be a set. $(A, \{\|\cdot\|_n\})$ is a matrix normed space if and only if $(M_n(A); \|\cdot\|_n)$ is a normed space for each positive integer n and $\|XuY\| \leq \|X\|\,\|Y\|\,\|u\|_n$ holds for $X \in M_{k,n}(\mathbb{C})$, $u = (u_{ij}) \in M_n(A)$ and $Y \in M_{n,k}(\mathbb{C})$ and that $(A, \{\|\cdot\|_n\})$ is a matrix Banach space if and only if A is a Banach space and $(A, \{\|\cdot\|_n\})$ a matrix normed space.

Definition 2.2 ([11])**.** A function $d : A \times A \to [0, \infty)$. A is a set, is called a generalized metric on A if d satisfies:

(1) $d(u,v) = 0$ if and only if $u = v$;

(2) $d(u,v) = d(v,u)$ for all $u, v \in A$;

(3) $d(u,w) = d(u,v) + d(v,w)$ for all $u, v, w \in A$.

3 Undecic Functional Equation (1.1)

In this section, we study the undecic functional equation (1.1). Throughout this section let us consider X and Y be real vector spaces.

Theorem 3.1. *If a mapping $f : X \to Y$ satisfies the functional equation (1.1) for all $u, v \in X$, then $f(2u) = 2^{11} f(u)$ for all $u \in X$.*

Proof. Consider that f satisfies the functional equation (1.1). Replacing (u,v) by $(0,0)$ in $(1,1)$, one obtains $f(0) = 0$. Now replacing (u,v) by $(0,u)$ and $(u,-u)$ in (1.1) and adding the two resulting equations, we have

$$f(-u) = -f(u)\,. \tag{3.1}$$

It shows that f is an odd mapping. Replacing (u,v) by $(0,u)$ in (1.1), and using (3.1), one finds that

$$f(12u) - 10f(10u) + 44f(86u) - 110f(6u) + 165f(4u) - 39916932f(2u) = 0, \tag{3.2}$$

for all $u \in X$. Replacing (u,v) with $(6u,u)$ in (1.1), one finds

$$\begin{aligned} &f(12u) - 11f(11u) + 55f(10u) - 165f(9u) + 330f(8u) - 462f(7u) + 462f(6u) \\ &- 330f(5u) + 165f(4u) - 55f(3u) + 11f(2u) - 39916801f(u) = 0, \end{aligned} \tag{3.3}$$

for all $u \in X$. Subtracting equation (3.3) from (3.2), we have

$$\begin{aligned} &11f(11u) - 65f(10u) + 165f(9u) - 286f(8u) + 462f(7u) - 572f(6u) + 330f(5u) \\ &+ 55f(3u) - 39916932f(2u) + 39916801f(u) = 0, \end{aligned} \tag{3.4}$$

for all $u \in X$. Replacing (u,v) with $(5u,u)$ in (1.1), and multiplying by 11, one finds

$$\begin{aligned} &11f(11u) - 121f(10u) + 605f(9u) - 1815f(8u) + 3630f(7u) - 5082f(6u) \\ &+ 5082f(5u) - 3630f(4u) + 1815f(3u) - 605f(2u) - f(2u) - 439084679f(u) = 0, \end{aligned} \tag{3.5}$$

for all $u \in X$. Now, subtracting (3.5) from (3.4), we have

$$56f(10u) - 440f(9u) + 1529f(8u) - 3168f(7u) + 4510f(6u) - 4752f(5u) \\ + 3630f(4u) - 17160f(3u) - 39916338f(2u) + 479001480f(u) = 0, \tag{3.6}$$

for all $u \in X$. Replacing (u, v) with $(4u, u)$ in (1.1), and multiplying the resulting equation by 56, one finds

$$56f(10u) - 616f(9u) + 3080f(8u) - 9240f(7u) + 18480f(6u) - 25872f(5u) \\ - 18480f(3u) + 9240f(2u) - 2235343824f(u) = 0, \tag{3.7}$$

for all $u \in X$. Subtracting equation (3.7) from (3.6), we have

$$176f(9u) - 1551f(8u) + 6072f(7u) - 13970f(6u) + 21120f(5u) - 22242f(4u) \\ + 16720f(3u) - 39925578f(2u) + 2714345304f(u) = 0, \tag{3.8}$$

for all $u \in X$. Considering (u, v) as $(3u, u)$ in (1.1), and multiplying the resulting equation by 176, one gets

$$176f(9u) - 1936f(8u) + 9680f(7u) - 29040f(6u) + 58080f(5u) - 81312f(4u) \\ + 81312f(3u) - 57904f(2u) - 7025329696f(u) = 0, \tag{3.9}$$

for all $u \in X$. Subtracting equation (3.8) and (3.9), we obtain

$$385f(8u) - 3608f(7u) + 15070f(6u) - 36960f(5u) + 59070f(4u) - 64592f(3u) \\ - 39867674f(2u) + 9739675000f(u) = 0, \tag{3.10}$$

for all $u \in X$ considering (u, v) is $(2u, u)$ in (1.1), and multiplying the resulting equation by 385, one obtains.

$$385f(8u) - 4235f(7u) + 21175f(6u) - 63525f(5u) + 127050f(4u) \\ - 177485f(3u) + 173636f(2u) - 15368073875f(u) = 0, \tag{3.11}$$

for all $u \in X$. Subtracting equation (3.11) from (3.10), we have

$$627f(7u) - 6105f(6u) + 26564f(5u) - 67980f(4u) + 112893f(3u) \\ - 40041309f(2u) + 25107748875f(u) = 0, \tag{3.12}$$

for all $u \in X$. Replacing (u, v) multiplying the (u, u) in (1.1), and multiplying the resulting equation by 627, one obtains

$$627f(7u) - 6897f(6u) + 34485f(5u) - 102828f(4u) + 200013f(3u)$$

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$$- 255189f(2u) - 25027647381f(u) = 0, \tag{3.13}$$

for all $u \in X$. Subtracting equation (3.13) from (3.12), we have

$$\begin{aligned} &792f(6u) - 7920f(5u) + 34848f(u) - 87120f(3u) - 39786120f(2u) \\ &+ 50135396256f(u) = 0, \end{aligned} \tag{3.14}$$

for all $u \in X$. Considering (u, v) as $(0, u)$ in (1.1), and multiplying the resulting equation by 792, one finds

$$\begin{aligned} &792f(6u) - 7920f(5u) + 34848f(4u) - 87120f(3u) + 130680f(2u) \\ &- 31614210144f(u) = 0, \end{aligned} \tag{3.15}$$

for all $u \in X$. Subtracting equations (3.14) and (3.15), we have

$$f(2u) = 2^{11}f(u4), \quad \text{for all } u \in X. \tag{3.16}$$

Hence $f : X \to Y$ is a undecic mapping This completes the proof. □

4 Stability of Undecic Functional Equation in Matrix Normed Spaces

In this section, we will prove the Ulam-Hyers stability for the functional equation (1.1) in matrix normed spaces by using the fixed point method. Throughont this section, we assume that $(A, \|\cdot\|_n)$ be is a matrix normed space, $(B, \|\cdot\|_n)$ be is a matrix Banach space and n is a fixed non-negative integer.

For a mapping $f : A \to B$, define $Df : A^2 \to B$ and $Df_n : M_n(A^2) \to M_n(B)$, for all $x, y \in X$ and all $u = [u_{ij}]$, $v = [v_{ij}] \in M_n(A)$.

Theorem 4.1. *Let $p = \pm 1$ be fixed and $\xi : A^2 \to [0, \infty)$ be a function such that there exists a $\delta < 11$ with*

$$\xi(x, y) \le 2^{11p}\delta\xi\left(\frac{x}{2^p}, \frac{y}{2^p}\right), \text{ for all } x, y \in A \tag{4.1}$$

consider $f : A \to B$ is a mapping satisfying

$$\|Df_n([u_{ij}], [v_{ij}])\| \le \sum_{i,j=1}^{n} \xi(u_{ij}, v_{ij}), \quad \text{for all } u = [u_{ij}],\ v = [v_{ij}] \in M_n(A). \tag{4.2}$$

Then there exists a unique undecic mapping $\tau : A \to B$ *such that*

$$\|f_n([u_{ij}]) - \tau_n([v_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{\delta^{\frac{1-p}{2}}}{2''(1-\delta)}\bar{\xi}(u_{ij}), \text{ for all } u = [u_{ij}] \in M_n(A), \tag{4.3}$$

where

$$\bar{\xi}(u_{ij}) = \frac{1}{11!}[\xi(0, 2u_{ij}) + \xi(6u_{ij}, u_{ij}) + 11\xi(5u_{ij}, u_{ij}) + 56\xi(4u_{ij}, u_{ij}) \\ + 176\xi(3u_{ij}, u_{ij}) + 385\xi(2u_{ij}, u_{ij}) + 627\xi(u_{ij}, u_{ij}) + 792\xi(0, u_{ij})].$$

Proof. Putting $n = 1$ in (4.2), we have

$$\|Df(x, y)\| \leq \xi(x, y). \tag{4.4}$$

Replacing (x, y) by $(0, 2x)$ and from (4.4), we obtain

$$\|f(12x) - 10f(10x) + 44f(8x) - 110f(6x) + 165f(4x) - 39916932f(2x)\| \\ \leq \psi(0, 2x), \tag{4.5}$$

for all $x \in A$. Replacing (x, y) by $(6x, x)$ in (4.4), we obtain

$$\|f(12x) - 11f(11x) + 55f(10x) - 165f(9x) + 330f(8x) - 462f(7x) \\ + 462f(6x) - 330f(5x) + 165f(4x) - 55f(3x) + 11f(2x) - 39916801f(x)\| \\ \leq \xi(6x, x), \tag{4.6}$$

for all $x \in A$. It follows from (4.5) and (4.6), we get

$$\|11f(11x) - 65f(10x) + 165f(9x) - 286f(8x) + 462f(7x) - 572f(6x) \\ + 330f(5x) + 55f(3x) - 39916943f(2x) + 39916801f(x)\| \\ \leq \xi(0, 2x) + \xi(6x, x), \tag{4.7}$$

for all $x \in A$. Changing (x, y) by $(5x, x)$ in (4.4), and multiplying by 11, we obtain

$$\|11f(11x) - 121f(10x) + 605f(9x) - 1815f(8x) + 3630f(7x) - 5082f(6x) \\ + 5082f(5x) - 3630f(4x) + 1815f(3x) - 605f(2x) - 439084679f(x)\| \\ \leq 11\xi(5x, x) \tag{4.8}$$

for all $x \in A$. It follows from (4.7) and (4.8), we find

$$\|56f(10x) - 440f(9x) + 1529f(8x) - 3168f(7x) + 4510f(6x) - 4752f(5x)$$

$$+ 3630f(4x) - 1760f(3x) - 39916338f(2x) + 479001480f(x)\|$$
$$\leq \xi(0, 2x) + \xi(6x, x) + 11\xi(5x, x), \tag{4.9}$$

for all $x \in A$. Replacing (x, y) by $(4x, x)$ in (4.4), and multiplying the resulting equation by 56, we have

$$\|56f(10x) - 616f(9x) + 3080f(8x) - 9240f(7x) + 18480f(6x) - 25872f(5x)$$
$$- 18480f(3x) + 9240f(2x) - 2235343824f(x)\| \leq 56\xi(4x, x), \tag{4.10}$$

for all $x \in A$. It follows from (4.9) and (4.10), we arrive at

$$\|176f(9x) - 1551f(8x) + 6072f(7x) - 13970f(6x) + 21120f(5x)$$
$$- 22242f(4x) + 16720f(3x) - 39925578f(2x) + 2714345304f(x)\|$$
$$\leq \xi(0, 2x) + \xi(6x, x) + 11\xi(5x, x) + 56\xi(4x, x) \tag{4.11}$$

for all $x \in A$. Replacing (x, y) by $(3x, x)$ in (4.4), and multiplying by 176, we have

$$\|176f(9x) - 1936f(8x) + 9680f(7x) - 29040f(6x) + 58080f(5x)$$
$$- 81312f(4x) + 81312f(3x) - 57904f(2x) - 7025329696f(x)\|$$
$$\leq 176\xi(3x, x) \tag{4.12}$$

for all $x \in A$. It follows from (4.11) and (4.12), we arive at

$$\|385f(8x) - 3608f(7x) + 15070f(6x) - 36960f(5x) + 59070f(4x)$$
$$- 64592f(3x) - 39867674f(2x) + 9739675000f(x)\|$$
$$\leq \xi(0, 2x) + \xi(6x, x) + 11\xi(5x, x) + 56\xi(4x, x) + 176\xi(3x, x), \tag{4.13}$$

for all $x \in A$. Replacing (x, y) by $(2x, x)$ in (4.4), and multiplying by 385, we arrive at

$$\|385f(8x) - 4235f(7x) + 21175f(6x) - 63525f(5x) + 127050f(4x)$$
$$- 177485f(3x) + 173636f(2x) - 15368073875f(x)\| \leq 385\xi(2x, x) \tag{4.14}$$

for all $x \in A$. It follows from (4.13) and (4.14), we obtain

$$\|627f(7x) - 6105f(6x) + 26564f(5x) - 67980f(4x) + 112893f(3x)$$
$$- 40041309f(2x) + 25107748875f(x)\|$$
$$\leq \xi(0, 2x) + \xi(6x, x) + 11\xi(5x, x) + 56\xi(4x, x) + 176\xi(3x, x) + 38\xi(2x, x)\|, \tag{4.15}$$

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for all $x \in A$. Replacing (x, y) by (x, x) in (4.4), and multiplying the resulting equation by 627, we get

$$\begin{aligned} &\|627f(7x) - 6897f(6x) + 34485f(5x) - 102828f(4x) + 200013f(3x) \\ &- 255189f(2x) - 25027647381f(x)\| \leq 627\xi(x, x), \end{aligned} \tag{4.16}$$

for all $x \in A$. It follows from (4.15) and (4.16), we arrive at

$$\begin{aligned} &\|792f(6x) - 7920f(5x) + 34848f(4x) - 87120f(3x) - 39786120f(2x) \\ &+ 50135396256f(x)\| \\ &\quad \leq \xi(0, 2x) + \xi(6x, x) + 11\xi(5x, x) + 56\xi(4x, x) + 176\xi(3x, x) \\ &\qquad + 385\xi(2x, x) + 627\xi(x, x), \end{aligned} \tag{4.17}$$

for all $x \in A$. Replacing (x, y) by $(0, x)$ in (4.4), and multiplying by 792, we get

$$\begin{aligned} &\|792f(6x) - 7920f(5x) + 34848f(4x) - 87120f(3x) + 130680f(2x) \\ &- 31614210144f(x)\| \leq 792\xi(0, x) \end{aligned} \tag{4.18}$$

for all $x \in A$. It follows from (4.17) and (4.18), we get

$$\begin{aligned} &\| - 39655440f(2x) + 18521186112f(x)\| \\ &\quad \leq \xi(0, 2x) + \xi(6x, x) + 11\xi(5x, x) + 56\xi(4x, x) \\ &\qquad + 176\xi(3x, x) + 385\xi(2x, x) + 627\xi(x, x) + 792\xi(0, x) \end{aligned} \tag{4.19}$$

for all $x \in A$. From equation (4.19), we get

$$\begin{aligned} \| - f(2x) + 2^{11}f(x)\| \leq \frac{1}{11!}[&\xi(0, 2x) + \xi(6x, x) + 11\xi(5x, x) + 56\xi(4x, x) \\ &+ 176\xi(3x, x)+385\xi(2x, x)+627\xi(x, x)+792\xi(0, x)]. \end{aligned} \tag{4.20}$$

Therefore,

$$\|f(2x) - 2^{11}f(x)\| \leq \bar{\xi}(x), \quad \text{for all } x \in A. \tag{4.21}$$

Thus.

$$\left\| f(x) - \frac{1}{2^{11p}} f\left(2^p x\right) \right\| \leq \frac{\delta^{\left(\frac{1-p}{2}\right)}}{2^{11}} \bar{\xi}(x), \quad \forall\, x \in A \tag{4.22}$$

We consider the set $P = \{f : A \to B\}$ and introduce the generalized metric ψ on P as follows:

$$\psi(f, g) = \inf\{\lambda \in \mathbb{R}_+ : \|f(x) - g(x)\| \leq \lambda\bar{\xi}(x) \text{ for all } x \in A\},$$

it is easy to check that (P, ψ) is a complete generalized metric ([13]). Define the mapping $\Phi : P \to P$ by

$$\Phi f(x) = \frac{1}{2^{11p}} f(2^p x), \quad \text{for all } f \in P \text{ and } x \in A.$$

Let $f, g \in P$ and γ be an arbitrary constant with $\psi(f, g) = \gamma$. Then

$$\|f(x) - g(x)\| \leq \gamma \bar{\xi}(x), \quad \text{for all } x \in A$$

Using (4.1), we get

$$\begin{aligned} \|\Phi f(x) - \bar{\Phi} g(x)\| &= \left\| \frac{1}{2^{11p} f} f\left(2^p x\right) - \frac{1}{2^{11p}} g\left(2^p x\right) \right\| \\ &\leq \delta \gamma \bar{\xi}(x), \quad \text{for all } x \in A. \end{aligned}$$

Hence it satisfies that $\psi(\Phi f, \Phi g) \leq \delta\gamma$, that is, $\psi(\Phi f, \Phi g) \leq \delta \psi(f, g)$ for all $f, g \in P$.

It follows from (4.22) that

$$\psi(f, \Phi f) \leq \frac{\delta^{\left(\frac{1-p}{2}\right)}}{2^{11}}.$$

Therefore according to Theorem 2.2 in [2] there exists a mapping $\tau : A \to B$ satisfying:

1. τ is a unique fixed point of Φ in the set $Q = \{g \in P : \psi(f, g) < \infty\}$ which is satisfied

$$\tau(2^p x) = 2^{11p} \tau(x), \quad \text{for all } x \in A \tag{4.23}$$

In other words, there exists a λ satisfying

$$\|f(x) - g(x)\| \leq \lambda \bar{\xi}(x), \quad \text{for all } x \in A$$

2. $\psi(\Phi^k f, \tau) \to 0$ as $k \to \infty$. This implies that

$$\lim_{k \to \infty} \frac{1}{2^{11k} p} f(2^{kp} x) = \tau(x), \quad \forall\, x \in A$$

3. $\psi(f, \tau) \leq \frac{1}{1-\delta} \psi(f, \Phi f)$, which imples the inequality $\psi(f, \tau) \leq \frac{\delta^{\frac{1-p}{2}}}{2^{11}(1-\delta)}$.

So

$$\|f(x)-\tau(x)\| \leq \frac{\delta^{\frac{1-p}{2}}}{2^{11}(1-\delta)}\bar{\xi}(x) \quad \forall\, x \in A \tag{4.24}$$

It follows from (4.1) and (4.2) that

$$\begin{aligned}\|D\tau(x,y)\| &= \lim_{k\to\infty}\frac{1}{2^{11kp}}\|Df(2^{kp}x,2^{kp}y)\| \\ &\leq \lim_{k\to\infty}\frac{1}{2^{11kp}}\xi(2^{kp}x,2^{kp}y) \\ &\leq \lim_{k\to\infty}\frac{2^{kp}\delta^k}{2^{11kp}}\xi(x,y)=0,\end{aligned}$$

for all $x, y \in A$. Therefore, the mapping $\tau : A \to B$ is undecic mapping. By Lemma 2.1 in [12] and (4.24),

$$\begin{aligned}\|f_n([u_{ij}])-\tau_n([u_{ij}])\| &\leq \sum_{i,j=1}^{n}\|f(u_{ij})-\tau(u_{ij})\| \\ &\leq \sum_{ij=1}^{n}\frac{\delta^{1-p}}{2^{11}(1-\delta)}\bar{\xi}(u_{ij}), \quad \text{for all } u=[u_{ij}]\in M_n(A)\end{aligned}$$

where

$$\begin{aligned}\bar{\xi}(u_{ij}) = \frac{1}{11!}[&\xi(0,2u_{ij})+\xi(6u_{ij},u_{ij})+11\xi(5u_{ij},u_{ij})+56\xi(4u_{ij},u_{ij})+176\xi(34_{ij},\mu_{ij}) \\ &+385\xi(2u_{ij},u_{ij})+627\xi(u_{ij},u_{ij})+792\xi(0,u_{ij})].\end{aligned}$$

Thus $\tau : A \to B$ is a unique undecic mapping satisfying (4.3). □

Corollary 4.2. *Let $p = \pm 1$ be fixed and let r, s be positive real numbers with $r \neq 11$. Let $f : A \to B$ be a mapping such that*

$$\|Df_n([u_{ij}],[v_{ij}])\|_n \leq \sum_{i,j=1}^{n} s(\|u_{ij}\|^r+\|v_{ij}\|^r), \quad \textit{for all } u=[u_{ij}],\ v=[v_{ij}]\in M_n(A) \tag{4.25}$$

Then there exists a unique undecic mapping $\tau : A \to B$ such that

$$\|f_n([u_{ij}])-\tau_n([u_{ij}])\|_n \leq \sum_{i,j=1}^{n}\frac{s_t}{p(2^{11}-2^r)}\|u_{ij}\|^r \quad \textit{for all} \ \ u=[u_{ij}]\in M_n(A),$$

where

$$s_t = \frac{s}{11!}[792+627(2^r)+385(3^r)+176(4^r)+56(5^r)+11(6^r)+7^r].$$

Proof. The proof follows from Theorem 4.1 by taking $\xi(x,y) = s(\|x\|^r + \|y\|^r)$, for all $x, y \in A$. Then we can choose $\delta = 2^{p(r-11)}$, and obtain the required result. $\square$

Example 4.3. Let $\xi : R \to R$ be a functim defined by

$$\xi(u) = \begin{cases} su^{11}, & \text{if } |u| < 1 \\ s, & \text{otherwise} \end{cases}$$

where $s > 0$ is a constant, and define a function $f : R \to R$ by

$$f(u) = \sum_{n=0}^{\infty} \frac{\xi(2^n u)}{2^{11n}}$$

for all $u \in R$. Then f satisfies the inequality

$$\begin{aligned} &\|f(u+6v) - 11f(u+5v) + 55f(u+4v) - 165f(u+3v) + 330f(u+2v) \\ &- 462f(u+v) + 462f(u) - 330f(u-v) + 165f(u-2v) - 55f(u-3v) \\ &+ 11f(u-4v) - f(u-5v) - 11!f(v)\| \\ &\quad \le \frac{(39918848)}{2047} s(2048)^2(|u|^{11} + |v|^{11}) \end{aligned} \tag{4.26}$$

for all $u, v \in R$. Then there do not exists a undecic mapping $\tau : R \to R$ and a constant $\eta > 0$ such that

$$|f(u) - \tau(u)| \le \eta |u|^{11}, \quad \text{for all } u \in R \tag{4.27}$$

Proof. It is easy to see that f satisfies (4.26).
If $u = v = 0$, then (4.26) is trivial.
If $|x|^{11} + |y|^{11} \ge \frac{1}{2^{11}}$, then LHS of (4.26) is less than $\frac{(39918848)(2048)}{2047} s$.
Suppose that $0 < |u|^{11} + |v|^{11} < \frac{1}{2^{11}}$, then there exists a non-negative integer k such that

$$\frac{1}{2^{11(k+1)}} \le |u|^{11} + |v|^{11} < \frac{1}{2^{11k}}, \tag{4.28}$$

so that $2^{11(k-1)}u^{11} < \frac{1}{2^{11}}$, $2^{11(k-1)}v^{11} < \frac{1}{2^{11}}$, and

$$\begin{aligned} &2^n(u), 2^n(v), 2^n(u+6v), 2^n(u+5v), 2^n(u+4v), 2^n(u+3v), 2^n(u+2v), \\ &2^n(u+v), 2^n(u-v), 2^n(u-2v), 2^n(u-3v), 2^n(u-4v), 2^n(u-5v), \\ &\qquad\qquad\qquad\qquad \text{for all } n = 0, 1, 2, \ldots, k-1. \end{aligned}$$

Hence

$$\xi(2^n(u+6v) - 11\xi(2^n(u+5v)) + 55\xi(2^n(u+4v)) - 165\xi(2^n(u+3v))$$
$$+ 330\xi(2^n(u+2v)) - 462\xi(2^n(u+v)) + 462\xi(2^n u) - 330\xi(2^n(u-v))$$
$$+ 165\xi(2^n(u-2v)) - 55\xi(2^n(u-3v)) + 11\xi(2^n(u-4v))$$
$$- \xi(2^n(u-5v)) - 11!\xi(2^n v) = 0,$$

for $n = 0, 1, 2, \ldots, k-1$. From the definition of f and (4.28), we get

$$|f(u+6v) - 11f(u+5v) + 55f(u+4v) - 165f(u+3v) + 330f(u+2v)$$
$$- 462f(u+v) + 462f(u) - 336f(u-v) + 165f(u-2v) - 55f(u-3v)$$
$$+ 11f(u-4v) - f(u-5v) - 11!f(v)|$$
$$\leq \sum_{n=0}^{\infty} \frac{1}{2^{11n}} |\xi(2^n(u+6v) - 11\xi(2^n(u+5u)) + 55\xi(2^n(u+4v)) - 165\xi(2^n(u+3v))$$
$$+ 330\xi(2^n(4+2v)) - 462\xi(2^n(u+v)) + 462\xi(2^n u) - 330\xi(2^n(u-v))$$
$$+ 165\xi(2^n(u-2v)) - 55\xi(2^n(u-3v)) + 11\xi(2^n(u-4v)) - \xi(2^n(u-5v)) - 11!\xi(2^n(y))$$
$$\leq \sum_{n=k}^{\infty} \frac{(39918848)}{2^{11n}} s$$
$$= \frac{(2048)(39918848)s}{2^{11k} \cdot 2047}$$
$$\leq \frac{(39918848)(2048)^2}{2047} s(|u|^{11} + |v|^{11}).$$

Thus f satisfies (4.26) for all $u, v \in R$ with $0 < |u|^{11} + |v|^{11} < \frac{1}{2^{11}}$.

Now, we claim that the undecic functional equation (1.1) is not stable for $r = 11$ in Corollary 4.2. Consider that there exists a undecic mapping $\tau : R \to R$ and a constant $\eta > 0$ satisfying (4.27). Then there exists a constant $t \in R$. So we get the following inequality

$$|f(u)| \leq (\eta + |t|)|u|^{11}, \tag{4.29}$$

Let $q \in N$ with $q\varepsilon > \eta + |t|$. If $u \in \left(0, \frac{1}{2^{q-1}}\right)$, then $2^n u \in (0,1)$ for all $n = 0, 1, 2, \ldots, q-1$ and for this case we find

$$f(u) = \sum_{n=0}^{\infty} \frac{\xi(2^n u)}{2^{11n}}$$
$$\geq \sum_{n=0}^{q-1} \frac{s(2^n u)^{11}}{2^{11n}}$$

$$= qsu^{11} > (\eta + |t|)|u|^{11}$$

which is a contradiction to (4.29).

Thus the undecic functional equation (1.1) is not stable for $r = 11$.

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