Geochromatic number of cubic graphs of order up to 12

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Abstract: *A set $S \subseteq V$ (G) is called a geodetic set if every vertex of G lies on a shortest path $u - v$ for some $u, v \in S$, the minimum cardinality among all geodetic sets is called geodetic number and is denoted by $g_n(G)$. A set $C \subseteq V$ (G) is called a chromatic set if C contains all vertices of different color classes in G, the minimum cardinality among all chromatic sets is called the chromatic number and is denoted by $x(G)$. A geochromatic set $S_c \subseteq V$ (G) is both a geodetic set and a chromatic set. The geochromatic number x_{gc}(G) of G is the minimum cardinality among all geochromatic sets of G. In this paper we determine the geochromatic number for cubic graphs of order up to 12 vertices.*

Keywords: **Geodetic sets, Chromatic sets, Geochromatic sets, Cubic graphs.**
Mathematics Subject Classification*[2010] Primary 05C51, 53C22, 58E10.*

1. Introduction

Cubic graphs are connected graphs having the property that each vertex has degree three. The origin of cubic graphs has been briefly explained in [8]. In an informal way, cubic graphs appeared in literature in [17], and in [20] more formally. Cubic graphs are natural models of some real world systems like, the interconnection networks of SIMD computers, several networks based on binary trees, cube-connected cycle networks [8], etc. Other than computer science we find applications in biology, chemistry in terms of Mosaic problems. In graph theory itself, cubic graphs play pivotal role in testing matching algorithms, map coloring problems, planarity testing problems, etc. Petersen graph is the best known example of a cubic, non-planar, 3-edge connected graph, that does not contain a Hamiltonian circuit. Fast generation of cubic graphs was done by Brinkmann [5]. Computer investigation of cubic graphs was conducted by Bussemaker et al. [6], for graphs of order upto 14. Non existence of cubic Distance Degree Injective (DDI) graphs of diameter 4, 5, 6 was discussed by Huilgol et al. [10], while Huilgol et al. [9] have constructed a self-centered, cubic DDI graph, to settle an open problem. PBIB-designs arising from diametral paths in cubic graphs was considered by Huilgol et al. [12]. PBIB-designs arising from geodetic sets in graphs was considered by Huilgol et al. [11]. Hence, the contributions of many researchers have enriched the area of cubic graphs, in terms of interesting properties, their extreme measures, with thier structures.

Recently, the study of geochromatic number was introduced by Samli et al. [18] and was further studied by Arul Mary [19], Huilgol et al. [13], [14]. As the name suggests, geochromatic number of a graph is a combination of chromaticity and geodeticity, hence acts as a double layered measure that covers all the vertices in a graph containing all color class representations. In a real world network model, a geochromatic set acts as the minimum number of all kinds of facility (emergency service) centers to be located in such a way that every node in a network can be reached using shortest distance paths (geodesics) from these facility centers. Therefore, we envisage a wide range of applications in real world.

2. Definitions and Preliminary Results

All the terms undefined here are in the sense of Buckley and Harary [2]. Here we consider undirected, finite graphs without loops and multiple edges. For any graph G the set of vertices is denoted by $v(G)$ and the edge set by $E(G)$. The order and size of G are denoted by p and q respectively.

Let u and v be vertices of a connected graph G. A shortest $u - v$ path is also called a u,v - geodesic. The (shortest path) distance is defined as the length of a u, v geodesic in G and is denoted by $d_{G(u,v)}$ or $d(u, v)$.

The eccentricity of vertex v of a graph G is the maximum distance between v and any other vertex of G. The diameter of G, denoted by $diam(G)$ is the maximum eccentricity of vertices in G, whereas the radius is the minimum such eccentricity, denoted by $rad(G)$.

Definition 2.1. [7] The (geodesic) interval $I(u, v)$ between u and v is the set of all vertices on all shortest $u - v$ paths. Given a set $S \subseteq V(G)$, its geodetic closure $I[S]$ is the set of all vertices lying on some shortest path joining two vertices of S. That is,

$$I[S] = v \in V(G) : v \in I(x, y), x, y \in S = \bigcup_{x,y} I(x, y).$$

A set $S \subseteq V(G)$, is called a geodetic set in G if $I[S] = V(G)$; that is, every vertex in G lies on some geodesic between two vertices from S. The geodetic number $g_n(G)$ of a graph G is the minimum cardinality of a geodetic set in G.

Note: The geodetic number of disconnected graphs is the sum of geodetic number of its components.

Definition 2.2. [8] An undirected graph $G = (V, E)$ is d-regular if each vertex $v \in V$ has exactly degree d. A cubic graph is a 3-regular graph.

Definition 2.3. [16] A n-vertex coloring of G is an assignment of n colors *1, 2, 3,....n* to the vertices of G: The coloring is proper if no two distinct adjacent vertices have the same color. If $x(G) = n$, then G is said to be n-chromatic where $n \le p$.

Definition 2.4. [16] A set $C \subseteq V(G)$ is called chromatic set if C contains all n vertices of distinct colors in G. Chromatic number of G is the minimum cardinality among all chromatic sets of G. That is, $x(G) = \{\min |c_i| / c_i$ is a chromatic set of $G\}$.

Definition 2.5. [18] A set s_c of vertices in G is said to be geochromatic set, if S_C is both geodetic and chromatic set. The minimum cardinality of a geochromatic set of G is its geochromatic number *(GCN)* and is denoted by $x_{gc}(G)$. A geochromatic set of size $x_{gc}(G)$ is said to be x_{gc}-set.

The following results are used.

Theorem 2.1. [3] Every geodetic set of a graph contains its extreme vertices.

Theorem 2.2. [7] If G is a non-trivial connected graph of order p and diameter d, then $g_n(G) \le p - d + 1$.

Theorem 2.3. [1] If every chromatic set of a graph G contains k vertices, then G has k vertices of degree at least $k - 1$.

Theorem 2.4. [16] Every minimum chromatic set of a graph G contains at most $(\Delta(G) + 1)$ vertices, where $\Delta(G)$ denotes the maximum degree of a graph G.

Theorem 2.5. [4] For any graph G, the chromatic number is at most one greater than its maximum degree, that is,

$$x(G) \le 1 + \Delta(G).$$

Theorem 2.6. [1] A graph is bi-colorable if and only if it has no odd cycles.

Theorem 2.7. [15] Every cubic graph is vertex colorable with three colors, except for tetrahedron that requires four colors.

3. Cubic graphs of order at most 12

As noted earlier, a cubic graph is 3-regular graph. There exists only one cubic graph of order 4 namely, the complete graph K_4, 2 cubic graphs of order 6, 5 cubic graphs of order 8, 21 cubic graphs of order 10 and 85 cubic graphs of order 12 by using the list provided by Bussemaker et al. [6]. The present study aimed to obtain the shortest paths in a cubic graph which covers all vertices in that graph containing all color class representation which is nothing but, the geochromatic number. Here we have calculated and tabulated all cubic graphs of order up to 12 their geodetic number, chromatic number and geochromatic number. These are all calculated by applying definitions and existing results. A simple program is written incorporating all conditions, the output of the same is tabulated in each case. We use the following notations: *d, r,* $g_n(G), x(G)$ *and* $x_{gc} = (GCN)$ denote diameter, radius, geodetic number, chromatic number, and geochromatic number of a graph, respectively.

Remark 1. The only cubic graph on *4* vertices is the complete graph, *K4*. We know that$g_n(k4) = 4$ and $x(K_4) = 4$ giving $x_{gc}(K_4) = 4$.

3.1. Cubic graphs with 6 vertices:

There exist 2 cubic graphs on 6 vertices [6] as given in Figure 1 below. Note that, both of them are self centered graphs, with geodetic number of each equal to 2. It is easy to observe that $x\,(G_1) = 3$. Hence, any chromatic set can be viewed as a geodetic as well as a geochromatic set, implying *GCN(G1)* = $x_{gc}(G_1) = 3$. The graph $G_2 = K_{3,3}$. Here also a geodetic set is not chromatic set as vertices at distance 2 receive the same color and hence vertices of a geodetic set lie in the same color class. So we add a vertex from other color class to make it a geochromatic set.

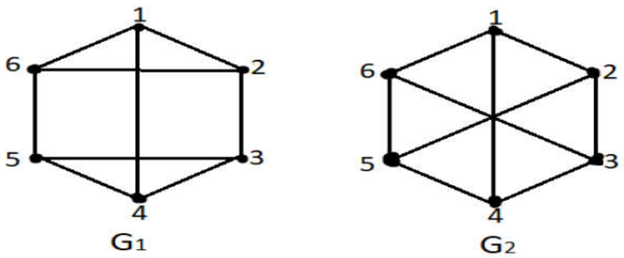

FIGURE 1. Cubic graphs with 6 vertices.

3.2. Cubic graphs with 8 vertices.

There are 5 cubic graphs on 8 vertices [6] as shown in Figure 2. For each one of these we have determined geochromatic number by simple calculation and shown in Table 1 below.

Table 1. Cubic Graphs on 8 vertices.

Graphs	d	r	$g_n(G)$	$x(G)$	$x_{gcd}(G)$
G_1	3	3	2	3	3
G_2	2	2	4	3	4
G_3	2	2	4	3	4
G_4	3	2	2	3	3
G_5	3	3	2	2	2

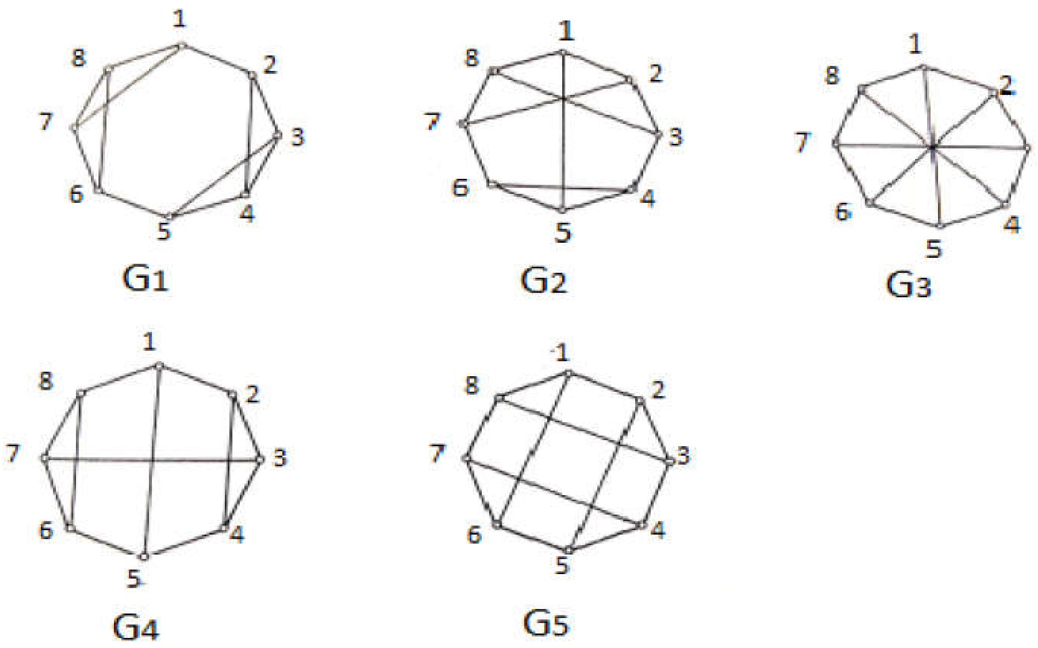

FIGURE 2. Cubic graphs of order 8.

3.3. Cubic graphs with 10 vertices.

There are 21 cubic graphs on 10 vertices [6] as shown in Figure 3 below. 14 of these graphs are self-centered graphs and 7 are not. Three graphs namely G_{14}, G_{16}, G_{20} are bipartite and the rest 3 colorable. We have given geodetic number and geochromatic number of each of them as in Table 2 below.

Table 2. Cubic Graphs on 10 vertices.

Graphs	d	r	$g_n(G)$	$\chi(G)$	$\chi_{gd}(G)$
G_1	5	3	4	3	4
G_2	3	3	3	3	4
G_3	3	3	3	3	3
G_4	4	2	4	3	5
G_5	3	3	3	3	3
G_6	3	3	3	3	3
G_7	4	3	3	3	3
G_8	3	3	3	3	3
G_9	3	2	3	3	3
G_{10}	3	3	3	3	3
G_{11}	3	2	4	3	3
G_{12}	3	2	3	3	3
G_{13}	3	3	3	3	3
G_{14}	3	3	3	2	3
G_{15}	3	3	3	3	3
G_{16}	3	3	3	2	3
G_{17}	3	2	4	3	4
G_{18}	3	3	3	3	3
G_{19}	3	3	3	3	3
G_{20}	3	3	3	2	3
G_{21}	2	2	4	3	4

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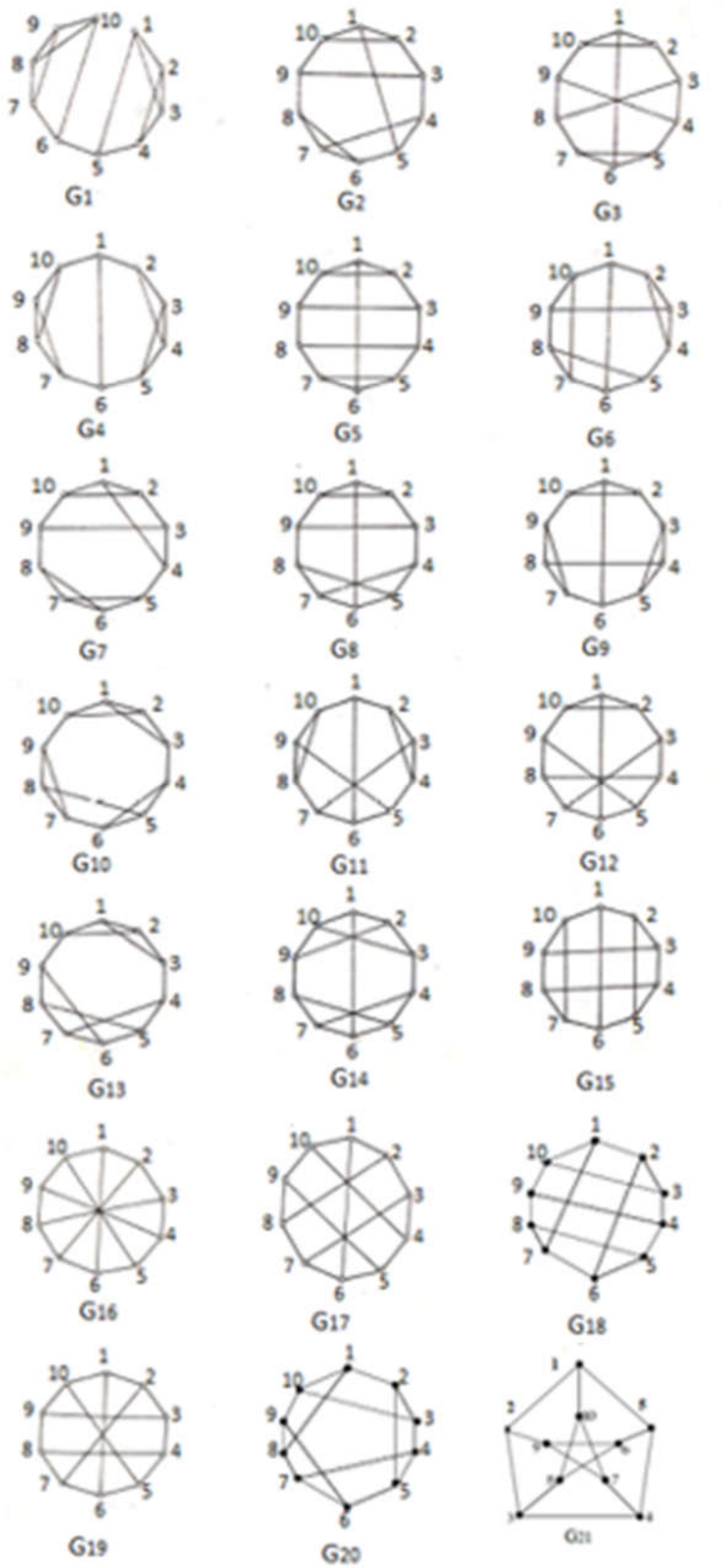

FIGURE 3. Cubic graphs on 10 vertices.

3.4. Cubic graphs with 12 vertices. There are 85 cubic graphs on 12 vertices [6] which are given in Figure 4. Table 3 gives a list of these 85 graphs and their geodetic, chromatic, geochromatic number along with their radius and diameter. Here 40 graphs are self-centered and the remaining 45 are not. Only 5 graphs are bipartite, namely, G_{11}, G_{46}, G_{68}, G_{75}, G_{81} and the remaining 80 graphs have their chromatic number 3.

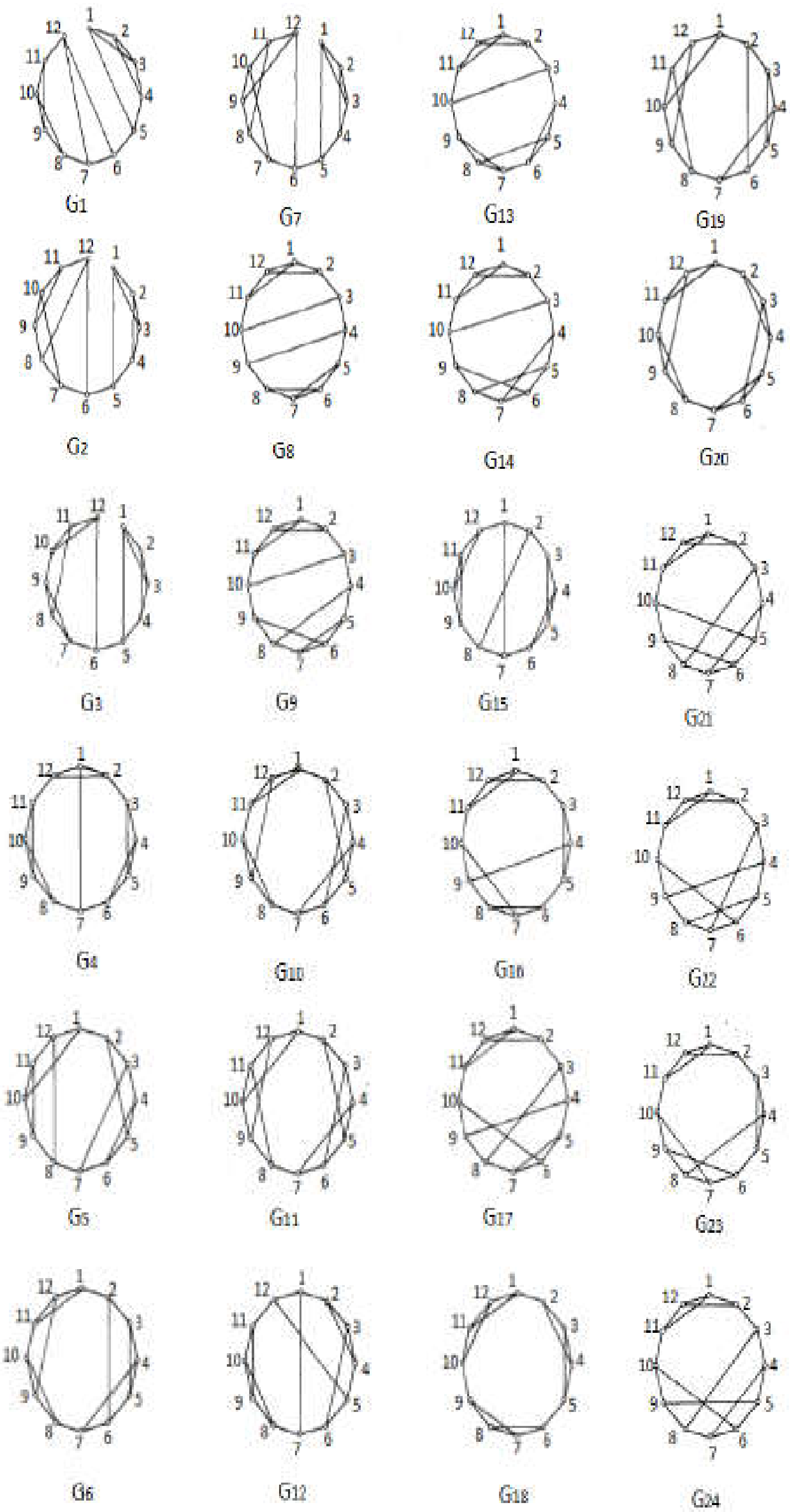

FIGURE 4. Cubic graphs on 12 vertices.

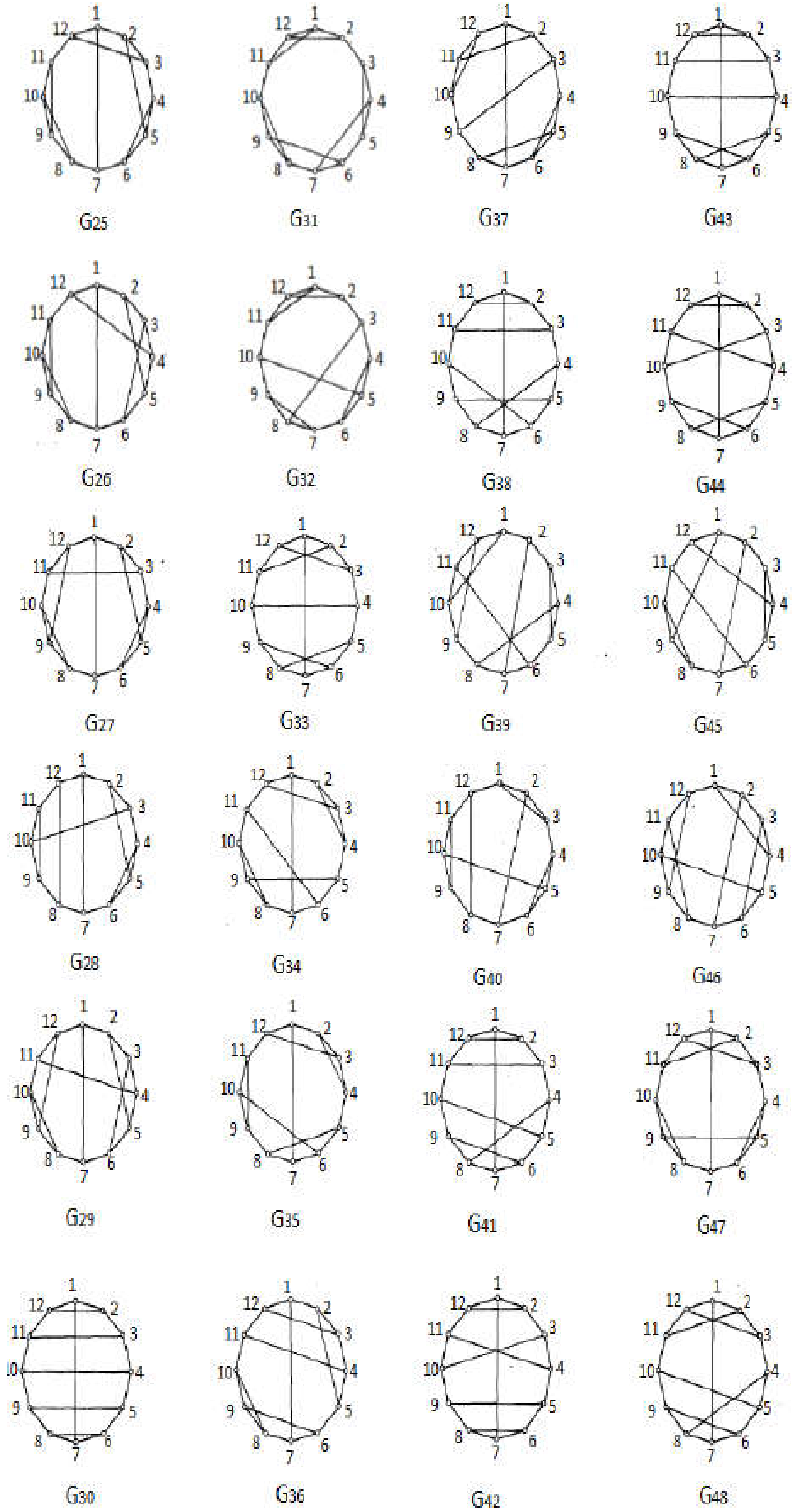

FIGURE 4. Cubic graphs on 12 vertices.

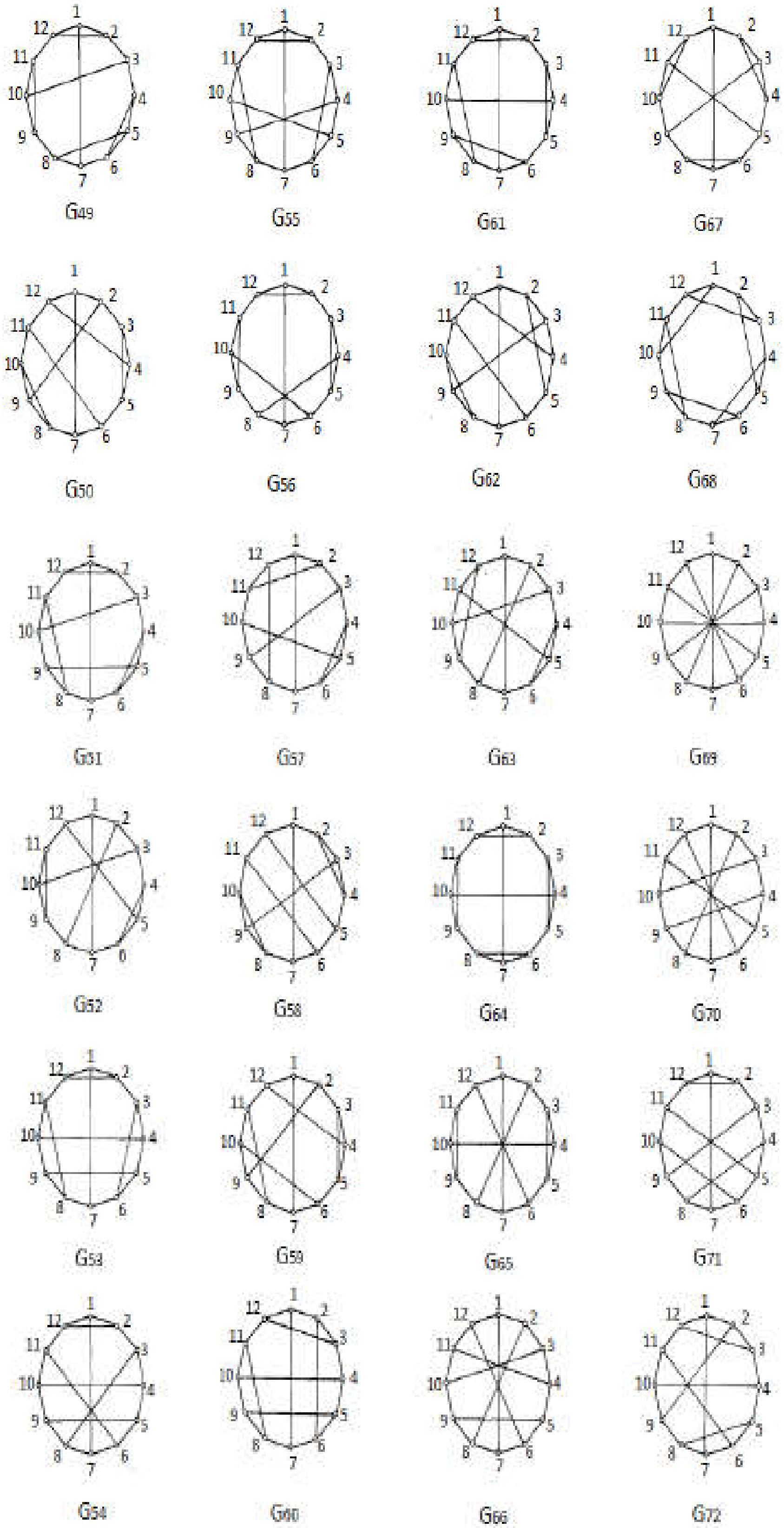

FIGURE 4. Cubic graphs on 12 vertices.

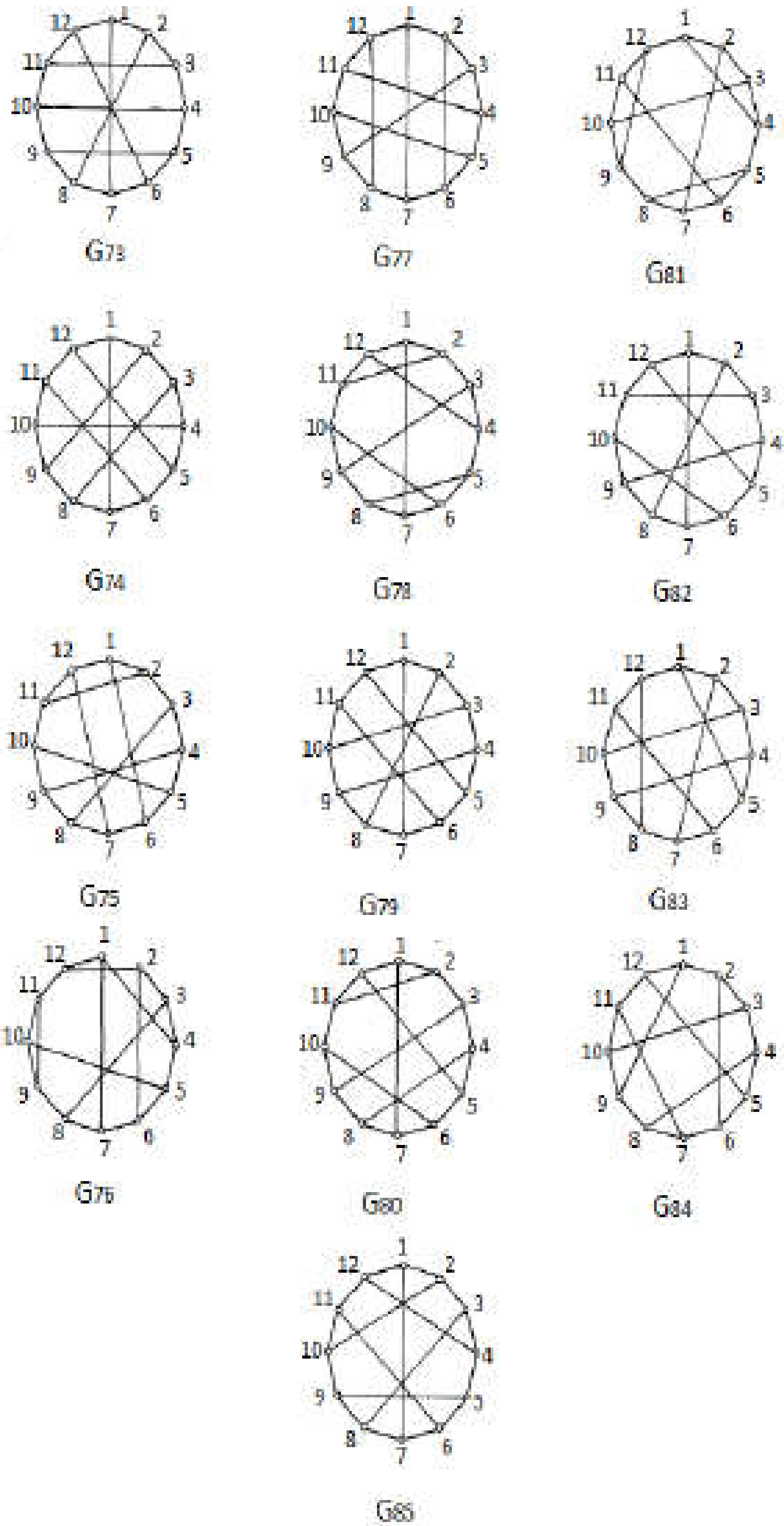

FIGURE 4. Cubic graphs on 12 vertices.

Table 3. Cubic Graphs on 12 vertices.

Graphs	d	r	$g_n(G)$	$\chi(G)$	$\chi_{gd}(G)$
G_1	6	3	4	3	4
G_2	6	3	4	3	4
G_3	5	3	4	3	4
G_4	4	3	3	3	3
G_5	5	3	2	3	3
G_6	4	3	3	3	3
G_7	5	3	5	3	5
G_8	5	3	2	3	3
G_9	5	3	3	3	3
G_{10}	4	4	2	3	3
G_{11}	4	4	2	2	3
G_{12}	4	3	3	3	3
G_{13}	4	3	4	3	4
G_{14}	4	3	4	3	4
G_{15}	5	3	4	3	4
G_{16}	4	3	3	3	3
G_{17}	4	3	3	3	3
G_{18}	4	4	3	3	3
G_{19}	4	3	3	3	3
G_{20}	4	4	2	3	3
G_{21}	4	3	3	3	3
G_{22}	4	3	4	3	4
G_{23}	4	4	2	3	3
G_{24}	4	3	3	3	4
G_{25}	4	3	3	3	3
G_{26}	4	3	3	3	3
G_{27}	4	3	3	3	3
G_{28}	4	3	3	3	3
G_{29}	4	3	3	3	3
G_{30}	4	3	3	3	3

Table 3. Cubic Graphs on 12 vertices.

Graphs	d	r	$g_n(G)$	$\chi(G)$	$\chi_{gd}(G)$
G_{31}	4	4	2	3	3
G_{32}	4	3	4	3	4
G_{33}	3	3	4	3	4
G_{34}	4	3	3	3	3
G_{35}	4	3	3	3	3
G_{36}	4	3	2	3	3
G_{37}	4	3	3	3	3
G_{38}	4	3	3	3	3
G_{39}	3	3	4	3	4
G_{40}	4	3	2	3	3
G_{41}	4	3	3	3	3
G_{42}	3	3	4	3	4
G_{43}	4	3	3	3	3
G_{44}	3	3	3	3	3
G_{45}	4	3	2	3	3
G_{46}	4	3	2	2	3
G_{47}	4	3	2	3	3
G_{48}	3	3	3	3	3
G_{49}	4	3	3	3	3
G_{50}	4	3	2	3	3
G_{51}	4	3	2	3	3
G_{52}	3	3	3	3	3
G_{53}	4	3	2	3	3
G_{54}	3	3	4	3	4
G_{55}	3	3	3	3	3
G_{56}	3	3	3	3	3
G_{57}	4	3	2	3	3
G_{58}	3	3	4	3	4
G_{59}	3	3	4	3	4
G_{60}	3	3	4	3	4

Table 3. Cubic Graphs on 12 vertices.

Graphs	d	r	$g_n(G)$	$\chi(G)$	$\chi_{gc}(G)$
G_{61}	4	3	2	3	3
G_{62}	3	3	4	3	4
G_{63}	3	3	4	3	4
G_{64}	3	3	3	3	3
G_{65}	3	3	3	3	3
G_{66}	3	3	4	3	4
G_{67}	3	3	3	3	3
G_{68}	4	4	2	2	3
G_{69}	3	3	3	3	3
G_{70}	3	3	4	3	4
G_{71}	3	3	3	3	3
G_{72}	3	3	4	3	4
G_{73}	3	3	3	3	3
G_{74}	3	3	4	3	4
G_{75}	4	3	2	2	3
G_{76}	3	3	3	3	3
G_{77}	3	3	4	3	4
G_{78}	3	3	4	3	4
G_{79}	3	3	4	3	4
G_{80}	3	3	3	3	3
G_{81}	3	3	4	2	4
G_{82}	3	3	4	3	4
G_{83}	3	3	4	3	4
G_{84}	3	3	4	3	4
G_{85}	3	3	3	3	3

4. Acknowledgments:

In this paper we have determined geodetic, chromatic and geochromatic number of cubic graphs of order up to 12. This can be continued for any cubic graph of higher order.

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