SOLUTION OF SYSTEM OF NON-LINEAR EQUATIONS BY NEWTON-RAPHSON METHOD

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Abstract: *We discuss the method of solving system of non-linear equations by Newton-Raphson method which is used to find a zero of the equation f(x)=0, in this paper. Few numerical examples are considered to exhibit the superiority of this method by comparing with Newton's method used for the solution of non-linear system of equations*

Keywords: **Newton-Raphson method, system of non-linear equations, Gauss-Seidel method.**

1. INTRODUCTION

Let us consider n real functions $f_1, f_2, ..., f_n$ which are functions of n real variables $x_1, x_2, ..., x_n$ .Then, the system of 'n' non-linear equations can be put in the form

$$\left.\begin{array}{l} f_1(x_1, x_2, ..., x_n) = 0 \\ f_2(x_1, x_2, ..., x_n) = 0 \\ . \\ . \\ . \\ f_n(x_1, x_2, ..., x_n) = 0 \end{array}\right\} ....(1.1)$$

The well known Newton's method for solving (1.1) is given by

$$X^{(n)} = X^{(n-1)} - J(X^{(n-1)})^{-1}.F(X^{(n-1)})$$
$$(n = 0,1,2,.......) \qquad ...(1.2)$$

where

$$X = (x_1, x_2, ...x_n)^T \quad .... \quad (1.3)$$
$$F = (f_1, f_2, ..., f_n)^T$$

and

$$J(X)^{-1} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(X) & \frac{\partial f_1}{\partial x_2}(X) & . & . & \frac{\partial f_1}{\partial x_n}(X) \\ \frac{\partial f_2}{\partial x_1}(X) & \frac{\partial f_2}{\partial x_2}(X) & . & . & \frac{\partial f_2}{\partial x_n}(X) \\ . & . & . & . & . \\ . & . & . & . & . \\ \frac{\partial f_n}{\partial x_1}(X) & \frac{\partial f_n}{\partial x_2}(X) & . & . & \frac{\partial f_n}{\partial x_n}(X) \end{bmatrix}^{-1} ....(1.4)$$

The Newton-Raphson method for the solution of a non-linear equation of a single variable

$$f(x) = 0 \quad .... \quad (1.5)$$

is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{........ (1.6)}$$

$$(n = 0,1,2,....)$$

Both the methods (1.2) and (1.6) converge quardratically. The former requires the calculation of the jacobian matrix and its inverse for each and every iteration which is quite time consuming and laborious.

In section 2, we discuss the method applying (1.6) to solve (1.1) and few examples are considered in the concluding section.

2. Newton-Raphson method for the solution of (1.1)

Let us denote the equations (1.1) in the form

$$\left.\begin{array}{l} F_1(x_1) = f_1(x_1, x_2, ..., x_n) = 0 \\ F_2(x_2) = f_2(x_1, x_2, ..., x_n) = 0 \\ . \\ . \\ . \\ F_n(x_n) = f_n(x_1, x_2, ..., x_n) = 0 \end{array}\right\} \text{............(2.1)}$$

where $F_1, F_2, ..., F_n$ are arrived while treating $x_2, x_3, ...,x_n$ as independent variables in f_1, treating $x_1, x_3, ..., x_n$ in f_2 and so on up to the same i.e., $x_1, x_2, ...,x_{n-1}$ in f_n.

Taking an initial guess $X^{(0)}$, we solve

$$F_1(X_1)=0 \text{ .... (2.2)}$$

for $x_1^{(1)}$ using (1.6)

With this new $x_1^{(1)}$ and other initial gausses for other variables, we solve

$$F_2(X_2)=0 \text{ .... (2.3)}$$

for $x_2^{(1)}$ using (1.6)

This process is continued till obtaining $x_n^{(1)}$ with $x_1^{(1)}, x_2^{(1)}, ..., x_{n-1}^{(1)}$ with the help of (1.6)

The whole above process is repeated till

$$\left\| X^{(n)} - X^{(n-1)} \right\| \to 0 \qquad (2.5)$$

up to the desired accuracy,

In general, the above process can be simply defined like Gauss-Seidel method, as

$$\left.\begin{array}{l} \hat{f}_1(x_1^{k+1}) = f_1(x_1^k, x_2^k, ..., x_n^k) = 0 \\ \hat{f}_2(x_2^{k+1}) = f_2(x_1^{k+1}, x_2^k, ..., x_n^k) = 0 \\ . \\ . \\ . \\ . \\ \hat{f}_n(x_n^{k+1}) = f_n(x_1^{k+1}, x_2^{k+1}, ..., x_{n-1}^{k+1}, x_n^k) = 0 \end{array}\right\} \text{......(2.6)}$$

$$(k = 0,1,2,...........)$$

by choosing a proper initial guess $X^{(0)}$.

3. NUMERICAL EXAMPLES

Example 1:

We consider the following system of non-linear equations which are solved in [9].

$$\left.\begin{aligned} f_1 &= x_1^2 - 10x_1 + x_2^2 + 8 = 0 \\ f_2 &= x_1x_2^2 + x_1 - 10x_2 + 8 = 0 \end{aligned}\right\} \text{.........} \quad (3.1)$$

whose exact solution of (3.1) is (1,1). Re-Writing the equations (3.1) as (2.2) and (2.3), we have

$$\left.\begin{aligned} F_1(\mathrm{x}_1) &= 0 \\ F_2(\mathrm{x}_2) &= 0 \end{aligned}\right\} \text{......} \quad (3.2)$$

Now, the gauss-seidel iterations for the solution (3.1) is given by

$$\left.\begin{aligned} F_1(\mathrm{x}_1^{n+1}) &= x_1^{(n)^2} - 10x_1^n + x_2^{(n)^2} + 8 = 0 \\ F_2(\mathrm{x}_2^{n+1}) &= x_1^{(n+1)}x_2^{(n)^2} + x_1^{(n+1)} - 10x_2^{(n)} + 8 = 0 \end{aligned}\right\} \text{.......}(3.3)$$

$$(\mathrm{n} = 0,1,2,.......)$$

For each 'n', we obtain the next approximation using the Newton-Raphson method (1.6).
Taking $x_1^{(0)} = x_2^{(0)} = 0.8$, we tabulate the successive approximations of the methods (1.2) and (3.3) along with the functional values upto six significant digits.

Table (1)

n	method	X_1,f_1	X_2,f_2
0	N.M	0.8,1.28	0.8,1.312
	N.R.M	0.8,1.28	0.8,1.312
1	N.M	0.98166470,0.13052171	0.99171726,0.29962399e^{-1}
	N.R.M	0.95238095,0.35200674	0.98426966,0.03233840
2	N.M	0.99901628,0.77648905e^{-2}	0.99994708,-0.15439731e^{-2}
	N.R.M	0.95586414,0.30986068e^{-1}	0.99894048,0.21432199e^{-3}
3	N.M	0.99999999,0.60000001e^{-7}	0.99999999,0.60000001e^{-7}

	N.R.M	0.9997334,0.19993355e^{-2}	0.99993323,0.10000588e^{-5}
4	N.M	1,0	1,0
	N.R.M	0.9999833,0.1252403e^{-3}	0.99999582,0.40157084e^{-7}
5	N.M	1,0	1,0
	N.R.M	0.99999896,0.78000012e^{-5}	0.99999974,0.60840222e^{-12}
6	N.M	1,0	1,0
	N.R.M	0.99999993,0.52000001e^{-6}	0.99999998,0.20000003e^{-7}
7	N.M	1,0	1,0
	N.R.M	1,0	1,0

Example 2:

Here, we consider a non-linear system of 3 equations in 3 unknowns which are also solved in [9] ,

$f_1 = 15x_1 + x_2^2 - 4x_3 = 13$

$f_2 = x_1^2 + 10x_2 - \exp(-x_3) = 11$

$f_3 = x_2^3 - 25x_3 = -22$

As done in example 1, we tabulate the results upto 8 significant digits taking $(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) = (0.86666667, 1.10000002, 0.88000000)$

Table (2)

n	method	X_1,f_1	X_2,f_2	X_3,f_3
0	N.M	0.86666667, -2.30999991	1.10000002, 0.33632841	0.88,1.33100007

	N.R.M	0.86666667, -2.30999991	1.10000002, 0.33632841	0.88,1.33100007
1	N.M	1.11364025, 0.65282166	1.011779832, -0.30096694e^{-2}	1.01887013, -2.43599583
	N.R.M	1.02066666, -0.31258542	1.03730225, 0.18110845e^{-1}	0.92464532, 0.28073959e^{-7}
2	N.M	1.04305725, 0.19531494	1.10314838, 0.71968388	0.91687004, 0.42071036
	N.R.M	1.04150569, -0.94989568e^{-2}	1.0311938, -0.31112809e^{-3}	0.92386124, 0.87299068e^{-7}
3	N.M	1.04230666, 0.30642881e^{-4}	1.03103551, -0.10790647e^{-4}	0.92440087, -0.13995718e^{-1}
	N.R.M	1.04213895, -0.15653784e^{-3}	1.03109296, -0.51484972e^{-5}	0.92384837,0.629238329e^{-8}
4	N.M	1.04221301, 0.12430793e^{-2}	1.03109806, 0.17260711e^{-3}	0.92377882, 0.17550226e^{-2}
	N.R.M	1.04214939,- 0.25210643e^{-5}	1.0310913,-0.71894526e^{-7}	0.92384816,-0.38199051e^{-7}
5	N.M	1.04215194,- 0.25162413e^{-7}	1.03109043,- 0.86508733e^{-7}	0.92385665, -0.21506302e^{-3}
	N.R.M	1.04214956,0.70702111e^{-8}	1.03109127,- 0.21533589e^{-7}	0.92384815,0.11611752e^{-6}
6	N.M	1.04215194,- 0.25162413e^{-7}	1.03109043,- 0.86508733e^{-7}	0.92385665,-0.21506302e^{-3}
	N.R.M	1.04214956, 0.70702111e^{-8}	1.03109127, -0.21533589e^{-7}	0.92384815, 0.11611752e^{-6}

Example 3:

Another non-linear system considered in [10] is given by

$$f_1 = 3x_1 - \cos(x_2 x_3) - \frac{1}{2} = 0$$
$$f_2 = x_1^2 - 81(x_2 + 0.1)^2 + \sin(x_3) + 1.06 = 0$$
$$f_3 = e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0$$

The approximations of Newton's method and N-R method are tabulated below upto 8 significant digits along with their functional values by taking
$(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) = (0.1, 0.1. - 0.1)$

Table (3)

n	method	X_1,f_1	X_2,f_2	X_3,f_3
0	N.M	0.1, -1.19995	0.1, -2.26983342	-0.1, 8.46624031
	N.R.M	0.1,-1.19995	0.1, -2.26983342	-0.1, 8.46624031
1	N.M	0.50003702,0.16259482e^{-3}	0.01946686, -0.34422082	-0.52152047, 0.36094149e^{-1}
	N.R.M	0.49998333,0.14069227e^{-3}	0.03735030, -0.71747142	-0.52288446, 0.37578177e^{-7}
2	N.M	0.50004593,0.13813588e^{-3}	0.00518859, -0.25857555e^{-1}	-0.52355711, 0.42542238e^{-2}
	N.R.M	0.49993643,-0.18713875e^{-3}	0.5103372e^{-2}, -0.84919966e^{-1}	-0.52368212, -0.33288583e^{-7}
3	N.M	0.50000034, 0.10200212e^{-5}	0.00001244, -0.20091856e^{-3}	-0.52359845, 0.42152563e^{-2}
	N.R.M	0.49999881, -0.3568038e^{-5}	0.11959016e^{-3}, -0.21196188e^{-2}	-0.52380653, 0.3183329e^{-7}
4	N.M	0.5,0	0,0.48482689e^{-8}	-0.52359877, 0.42150763e^{-2}
	N.R.M	0.5,0.16663559e^{-10}	-0.1102117e^{-4}, -0.42091018e^{-5}	-0.5238098, -0.13092983 e^{-7}

5	N.M	0.5,0	0, 0.48482689 e^{-8}	-0.52359877, 0.42150763e^{-2}
	N.R.M	0.5,0.17458479e^{-10}	-0.11280932e^{-4}, -0.92273231e^{-8}	-0.52380981, -0.83184759e^{-7}
6	N.M	0.5,0	0,0.48482689e^{-8}	-0.52359877, 0.42150763e^{-2}
	N.R.M	0.5,0.17459589e^{-10}	-0.1128128e^{-4},-0.35903591e^{-8}	-0.52380981,-0.83010757e^{-7}

CONCLUSION

As it is clear from the above tabulated results that the system of nonlinear equations can be solved easily with the method (1.6) rather than the much laborious one (1.2).

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