Symmetric K-Derivatives

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Abstract

Using a two variable kernel $K(s,t)$, a new kind of symmetric derivative is defined for a function so that it coincides with symmetric Laplace derivative by taking a particular kernel $K(s,t) = e^{-st}$. Some basic properties of this new derivative are studied.

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1 Introduction

Laplace derivative was introduced by Sevtic in [10]. Later many authors studied Laplace derivative in [4], [5], [7], [8], [9]. Symmetric Laplace derivative was introduced in [1] and it was shown that symmetric Laplace derivative is a generalization of symmetric d.l.V.P. derivative [2]. Also the properties of symmetric Laplace derivative are discussed there. A brief discussion on Laplace derivative and symmetric Laplace derivative can be found in the book of Mukhopadhyay [6]. In the definitions of both the derivatives author used e^{-st} as the Kernel. The definition of K-derivative is given in [4] using the kernel $K(s,t)$. In this article, we introduce symmetric K- derivative and show that symmetric K-derivative is a generalisation of symmetric d.l.V.P. derivative, also it is shown that symmetric K-derivative shares many properties of symmetric d.l.V.P. derivative. To prove the analogous properties of symmetric K- derivative we used same procedure of [1] with suitable modifications.

2 Definitions and Notations

Definition 2.1. *Let* $K : [0,\infty) \times [0,\infty) \to \mathbf{R}$ *be a positive continuous function which satisfies the following conditions:*

(i) $K(s,t)$ *has bounded partial derivative with respect to* t.
(ii) $\lim_{s\to\infty} s^{n+1}K(s,t) = 0,\ \lim_{s\to\infty} s^{n+1}\frac{\partial K}{\partial t} = 0$ *for all* n *and for any* t.
(iii) $\int_0^\delta K(s,t)t^p dt = p!s^{-p-1} + o(1)$ *as* $s \to \infty$.
We call $K(s,t)$ *a kernel function.*

Definition 2.2. *Let* f *be special Denjoy integrable function in some neighbourhood of* x, *then* f *is said to have n-th order right hand* K*-derivative at* x *if there are real numbers* $\alpha_0, \alpha_1, ..., \alpha_n$ *such that*

$$\lim_{s\to\infty} s^{n+1}\int_0^\delta K(s,t)\left[f(x+t) - \sum_{i=0}^{n}\frac{t^i}{i!}\alpha_i\right]dt = 0$$

for some kernel $K(s,t)$ *and a fixed* $\delta > 0$. α_n *is called the right hand n-th order* K*-derivative of* f *at* x *and is denoted by* $KD_n^+ f(x)$.
Similarly, the left hand n-th order K*-derivative of* f *at* x *can be defined and is denoted by* $KD_n^- f(x)$. f *is said to have n-th order left hand* K*-derivative at* x *if there are real numbers* $\beta_0, \beta_1, ..., \beta_n$ *such that*

$$\lim_{s\to\infty}(-1)^n s^{n+1}\int_0^\delta K(s,t)\left[f(x-t) - \sum_{i=0}^{n}\frac{(-t)^i}{i!}\beta_i\right]dt = 0$$

for some kernel $K(s,t)$ *and a fixed* $\delta > 0$. *Then* β_n *is called the left hand n-th order* K*-derivative of* f *at* x *and is denoted by* $KD_n^- f(x)$.
If $KD_i^+ f(x) = KD_i^- f(x)$ *for* $i = 0, 1, ..., n$ *then* f *is said to have n-th order left hand* K*-derivative at* x *and it is denoted by* $KD_n f(x)$.

Definition 2.3. *Let* f *be special Denjoy integrable function in some neighbourhood of* x, *then* f *is said to have symmetric* K*-derivative of order* $2n$ *at* x *if there are real numbers* $\alpha_0, \alpha_2, ..., \alpha_{2n}$ *such that*

$$\lim_{s\to\infty} s^{2n+1}\int_0^\delta K(s,t)\left[\frac{f(x+t)+f(x-t)}{2} - \sum_{i=0}^{n}\frac{t^{2i}}{(2i)!}\alpha_{2i}\right]dt = 0$$

for some kernel $K(s,t)$ *and a fixed* $\delta > 0$. *Then* α_{2n} *is called the symmetric* K*-derivative of* f *of order* $2n$ *at* x *and is denoted by* $SKD_{2n}f(x)$.
The symmetric K*-derivative of* f *of order* $(2n+1)$ *at* x *can be defined as,* f *is said to have the symmetric* K*-derivative of order* $(2n+1)$ *at* x *if there are real numbers* $\beta_1, ..., \beta_{2n+1}$ *such that*

$$\lim_{s\to\infty} s^{2n+2}\int_0^\delta K(s,t)\left[\frac{f(x+t)-f(x-t)}{2} - \sum_{i=0}^{n}\frac{t^{2i+1}}{(2i+1)!}\beta_{2i+1}\right]dt = 0$$

for some kernel $K(s,t)$ *and a fixed* $\delta > 0$. *Then* β_{2n+1} *is called the symmetric* K*-derivative of* f *of order* $(2n+1)$ *at* x *and is denoted by* $SKD_{2n+1}f(x)$.

Note 2.4. *The above definition is independent of* δ.

Proof: Let for $0 < \delta_1 < \delta_2$,

$$\lim_{s\to\infty} s^{2n+1}\int_0^{\delta_1} K(s,t)\left[\frac{f(x+t)+f(x-t)}{2} - \sum_{i=0}^{n}\frac{t^{2i}}{(2i)!}SKD_{2i}f(x)\right]dt$$

$$- \lim_{s\to\infty} s^{2n+1} \int_0^{\delta_2} K(s,t) \left[\frac{f(x+t)+f(x-t)}{2} - \sum_{i=0}^{n} \frac{t^{2i}}{(2i)!} SKD_{2i}f(x) \right] dt$$
$$= \lim_{s\to\infty} s^{2n+1} \int_{\delta_1}^{\delta_2} K(s,t) \left[\frac{f(x+t)+f(x-t)}{2} - \sum_{i=0}^{n} \frac{t^{2i}}{(2i)!} SKD_{2i}f(x) \right] dt$$
$$= \lim_{s\to\infty} s^{2n+1} K(s,\delta_2) \int_{\delta_1}^{\delta_2} \left[\frac{f(x+t)+f(x-t)}{2} - \sum_{i=0}^{n} \frac{t^{2i}}{(2i)!} SKD_{2i}f(x) \right] dt$$
$$- \lim_{s\to\infty} s^{2n+1} \int_{\delta_1}^{\delta_2} \frac{\partial K}{\partial t} \int_{\delta_1}^{t} \left[\frac{f(x+\xi)+f(x-\xi)}{2} - \sum_{i=0}^{n} \frac{\xi^{2i}}{(2i)!} SKD_{2i}f(x) \right] d\xi dt$$
$$= 0,$$

as $\left[\frac{f(x+\xi)+f(x-\xi)}{2} - \sum_{i=0}^{n} \frac{\xi^{2i}}{(2i)!} SKD_{2i}f(x) \right]$ is continuous for all n and for any t, given in Definition 2.1.

Analogously, it can be shown that

$$\lim_{s\to\infty} s^{2n+2} \int_0^{\delta_1} K(s,t) \left[\frac{f(x+t)-f(x-t)}{2} - \sum_{i=0}^{n-1} \frac{t^{2i+1}}{(2i+1)!} SKD_{2i+1}f(x) \right] dt$$
$$= \lim_{s\to\infty} s^{2n+2} \int_0^{\delta_2} K(s,t) \left[\frac{f(x+t)-f(x-t)}{2} - \sum_{i=0}^{n-1} \frac{t^{2i+1}}{(2i+1)!} SKD_{2i+1}f(x) \right] dt.$$

Definition 2.5. *A function f is said to be Smooth of order $(2n+2)$ w. r. t kernel $K(s,t)$ if $SKD_{2n}f(x)$ exists and*

$$\lim_{s\to\infty} s^{2n+2} \int_0^{\delta} \left[\frac{f(x+t)+f(x-t)}{2} - \sum_{i=0}^{n} [\frac{\xi^{2i}}{(2i)!} SKD_{2i}f(x) \right] = 0.$$

A function f is said to be Smooth of order $(2n+1)$ w. r. t kernel $K(s,t)$ if $SKD_{2n-1}f(x)$ exists and

$$\lim_{s\to\infty} s^{2n+1} \int_0^{\delta} \left[\frac{f(x+t)-f(x-t)}{2} - \sum_{i=0}^{n-1} \frac{\xi^{2i+1}}{(2i+1)!} SKD_{2i+1}f(x) \right] = 0.$$

Note 2.6. *i) If $K(s,t) = e^{-st}M(s,t)$ where $M(s,t)$ is a positive, continuous function and it satisfies the three conditions given in Definition 2. 1, then $K(s,t)$ is a kernel satisfying Definition 2. 1 and symmetric K-derivative of an integrable function f can be defined w. r. t. this kernel $K(s,t)$.*
ii) Symmetric Laplace derivative is a special type of symmetric K-derivative.

3 Properties of the Derivative

Lemma 3.1. *Let $g(t)$ is continuous in right neighborhood N of zero and if $g(t) \to 0$ as $t \to 0+$ then*

$$s \int_0^{\delta} K(s,t)g(t)dt \to 0 \text{ as } s \to \infty.$$

Proof: Let $\epsilon > 0$ be arbitrary. Since $g(t) \to 0$ as $t \to 0+$ then there exists $\delta_1 (> 0), \delta_1 \in N$ such that $|g(t)| < \epsilon$ for $0 < t < \delta_1$. Therefore,

$$\left| s \int_0^{\delta_1} K(s,t)g(t)dt \right| \leq \epsilon \left| s \int_0^{\delta_1} K(s,t)dt \right| \leq \epsilon.$$

Let $\delta_1 < \delta$ and $\delta \in N$ now as before,

$$s\int_{\delta_1}^{\delta} K(s,t)g(t)dt \to 0 \text{ as } s \to \infty.$$

Hence $\lim_{s\to\infty}\left|\int_0^{\delta} K(s,t)g(t)dt\right| < \epsilon$, and δ is independent of ϵ. Since ϵ is arbitrary our result is proved.

Remark 3.2. *Let $KD_nf(x)$ exists then $SKD_nf(x)$ exists with equal value. In fact*

$$SKD_nf(x) = \tfrac{KD_n^+f(x)+KD_n^-f(x)}{2}.$$

Theorem 3.3. *Let f:(a, b)→ **R** be integrable and $x_0 \in (a,b)$. If $SKD_{2n}f(x_0)$ exists with associated real numbers $\alpha_0, \alpha_2, ..., \alpha_{2n-2}$ then*

$$\alpha_{2i} = SKD_{2i}f(x_0)$$

for $i = 0, 1, ..., n-1$.

Proof: $SKD_{2n}f(x_0) = \lim_{s\to\infty} s^{2n+1}\int_0^{\delta} K(s,t)\left[\frac{f(x_0+t)+f(x_0-t)}{2} - f(x_0)\sum_{i=0}^{n-1}\frac{t^{2i}}{(2i)!}\alpha_{2i}\right]dt$

Let $1 \le m < n$ and $\epsilon(s)$ be such that $\epsilon(s) \to 0$ as $s \to \infty$ and

$$SKD_{2n}f(x_0) + \epsilon(s) = s^{2n+1}\int_0^{\delta} K(s,t)\left[\frac{f(x_0+t)+f(x_0-t)}{2} - f(x_0)\sum_{i=0}^{n-1}\frac{t^{2i}}{(2i)!}\alpha_{2i}\right]dt$$
$$= s^{2n+1}\int_0^{\delta} K(s,t)\left[\frac{f(x_0+t)+f(x_0-t)}{2} - f(x_0)\sum_{i=0}^{m-1}\frac{t^{2i}}{(2i)!}\alpha_{2i}\right]dt$$
$$- s^{2n+1}f(x_0)\int_0^{\delta} K(s,t)\left[\frac{t^{2m}}{(2m)!}\alpha_{2m}\right]dt - s^{2n+1}f(x_0)\int_0^{\delta} K(s,t)\left[\sum_{i=m+1}^{n-1}\frac{t^{2i}}{(2i)!}\alpha_{2i}\right]dt$$

Dividing both side by $s^{2(n-m)}$ and letting $s \to \infty$ by Lemma 3. 1 we get

$$SKD_{2m}f(x_0) - \alpha_{2m} = \lim_{s\to\infty} s^{2n+1}f(x_0)\int_0^{\delta} K(s,t)\left[\sum_{i=m+1}^{n-1}\frac{t^{2i}}{(2i)!}\alpha_{2i}\right]dt = 0.$$

Hence, $SKD_{2m}f(x_0) = \alpha_{2m}$.

For $\alpha_0 = f(x_0)$ we consider

$$SKD_2f(x_0) = \lim_{s\to\infty} s^3\int_0^{\delta} K(s,t)\left[\frac{f(x_0+t)+f(x_0-t)-2\alpha_0}{2}\right]dt$$

So,

$$SKD_2f(x_0) + \epsilon_1(s) = s^3\int_0^{\delta} K(s,t)\left[\frac{f(x_0+t)+f(x_0-t)-2\alpha_0}{2}\right]dt,$$

where $\epsilon_1(s) \to 0$ as $s \to \infty$. Thus,

$$\tfrac{SKD_2f(x_0)}{s^2} + \tfrac{\epsilon_1(s)}{s^2} = s\int_0^{\delta} K(s,t)\left[\frac{f(x_0+t)+f(x_0-t)-2\alpha_0}{2}\right]dt.$$

So,

$$s\int_0^\delta K(s,t)\left[\frac{f(x_0+t)+f(x_0-t)-2\alpha_0}{2}\right]dt \to 0 \text{ as } s\to\infty. \;.\;.\;.\;.\;.\;.\;(1)$$

Since f is continuous at x_0, we have $\left[\frac{f(x_0+t)+f(x_0-t)-2\alpha_0}{2}\right] \to 0$ as $s\to\infty$ and from Lemma 3. 1,

$$s\int_0^\delta K(s,t)\left[\frac{f(x_0+t)+f(x_0-t)-2\alpha_0}{2}\right]dt \to 0 \text{ as } s\to\infty. \;.\;.\;.\;.\;.\;(2)$$

Subtracting (1) from (2) we get

$$s\int_0^\delta K(s,t)\left[f(x_0)-\alpha_0\right]dt \to 0 \text{ as } s\to\infty.$$

Thus,

$$f(x_0)-\alpha_0 \to 0 \text{ as } s\to\infty.$$

Hence, $f(x_0)=\alpha_0$.

The proof for odd case is analogous.

Theorem 3.4. *Let f be a function integrable in [a, b] and $x \in$(a, b), then*

$$\underline{SD}_r f^+(x) \le \underline{SKD}_r^+ f(x) \le \overline{SKD}_r^+ f(x) \le \overline{SD}_r f^+(x),$$

where $\underline{SD}_r f(x)$ is symmetric lower d. l. V. P derivative of f of order r at x and $\overline{SD}_r f(x)$ is symmetric upper d. l. V. P derivative of f of order r at x.

Proof: Let $r=2n$ be even, we prove only the right inequality $\overline{SKD}_{2n}^+ f(x) \le \overline{SD}_{2n} f^+(x)$, the rest is similar.

If $\overline{SD}_{2n} f^+(x)=\infty$ the proof is clear.

Thus we suppose that $\overline{SD}_n f^+(x) < M$ for some $M>0$. So there exists $\delta>0$ satisfying

$$\left[\tfrac{f(x+t)+f(x-t)}{2} - \sum_{i=0}^{n-1}\frac{t^{2i}}{(2i)!}SD_{2i}f(x)\right] < M\tfrac{t^{2n}}{(2n)!} \text{ for } 0<t<\delta.$$

Hence,

$$s^{2n+1}\int_0^\delta K(s,t)\left[\frac{f(x+t)+f(x-t)}{2} - \sum_{i=0}^{n-1}\frac{t^{2i}}{(2i)!}f_{(2i)}(x)\right]dt < M\frac{s^{2n+1}}{(2n)!}\int_0^\delta K(s,t)t^{2n}dt.$$

Letting $s\to\infty$, we get $\overline{SKD}_{2n}f(x) \le M$.

Since M is arbitrary upper bound of $\overline{SDf}_{2n}^+(x)$, we have

$$\overline{SD}_{2n}^+ f(x) \le \overline{SKD} f_{(2n)}^+(x).$$

The proof for Odd case is similar.

Remark 3.5. *If $SDf^n(x)$ exists where $\underline{SD}f^r(x)$ is symmetric d. l. V. P derivative of f of order r at x then $SKD_n f(x)$ exists with equal value.*

The converse is not necessarily true. It can be shown with an example.

Let $f(x)=\sin\frac{1}{x^2}$ *if* $x>0$,

$-\sin\frac{1}{x^2}$ *if* $x<0$,

0 *if* $x=0$.

Then $SD_1 f(0)$ does not exist, as $\lim\limits_{t\to 0}\cos\dfrac{1}{t^2}$ *does not exist . But*

$$s^2\int_0^\delta e^{-st}\cos\frac{1}{t^2}dt \to 0 \ \textit{as}\ s\to\infty.$$

Then $SKD_1f(0)=0$ *with respect to kernel* $K(s,t)=e^{-st}$.

Lemma 3.6. *Let* f *is smooth of order two at* x *w. r. t kernel* $K(s,t)$ *then*

$$\overline{KD}_1^+f(x)=\overline{KD}_1^-f(x), \underline{KD}_1^+f(x)=\underline{KD}_1^-f(x).$$

Proof: $\lim_{s\to\infty} s^2\int_0^\delta K(s,t)\,[f(x+t)-f(x)]\,dt+\lim_{s\to\infty} s^2\int_0^\delta K(s,t)\,[f(x-t)-f(x)]\,dt$
$=\lim_{s\to\infty} s^2\int_0^\delta K(s,t)\,[f(x+t)+f(x-t)-2f(x)]\,dt=0,$
implies proof of the lemma.

Lemma 3.7. *Let* f *is maximum at* x *and smooth of order two at* x *w. r. t kernel* $K(s,t)$ *then* $KD_1f(x)$ *exists and* $KD_1f(x)=0$.

Proof: Since f is maximum at x , there exists $\delta>0$ such that

$$f(x+t)\le f(x), f(x-t)\le f(x) \text{ whenever } 0\le t<\delta.$$

Thus for any s,

$$\mathrm{s}^2\int_0^\delta K(s,t)\,[f(x+t)-f(x)]\,dt\le 0.$$

So,

$$\limsup_{s\to\infty} s^2\int_0^\delta K(s,t)\,[f(x+t)-f(x)]\,dt\le 0, \text{ implies } \overline{KD}_1^+f(x)\le 0.$$

Again for any s,

$$\mathrm{s}^2\int_0^\delta K(s,t)\,[f(x-t)-f(x)]\,dt\le 0\ .$$

Hence,

$$(-1)\mathrm{s}^2\int_0^\delta K(s,t)\,[f(x-t)-f(x)]\,dt\ge 0\ .$$

So,

$$\liminf_{s\to\infty}(-1)s^2\int_0^\delta K(s,t)\,[f(x-t)-f(x)]\,dt\ge 0, \text{ implies } \underline{\mathrm{KD}}_1^-f(x)\le 0.$$

Therefore, $\overline{KD}_1^+f(x\le 0\le \underline{KD}_1^-f(x)$.

By Lemma 3. 6, the proof is completed.

Theorem 3.8. *Let* $f:(a,b)\to \boldsymbol{R}$ *be such that*

(i) f *is continuous on* (a,b).

(ii)$\overline{SKD}_2f(x)\ge 0$ *for all* $x\in(a,b)/N$ *where* N *is a countable subset of* (a,b).

(iii) f *is smooth of order two on* (a,b) *w. r. t same kernel.*

Then f *is convex on* (a,b).

Proof: $\overline{SKD}_2 f(x) \geq 0$ for all $x \in (a,b)/N$ where N is a countable subset of (a,b), so by theorem 3. 4, $\overline{SD}_2 f(x) > 0$ for all $x \in (a,b) - N$ and also suppose that f is not convex on (a,b). So there is m_0 and c such that $g(x) = f(x) + m_0 x + c$ has a maximum at some point $z \in (a,b)$. For sufficiently small $\epsilon(> 0)$ every number $m \in (m_0 - \epsilon, m_0 + \epsilon)$ has again the same property. For each $m \in (m_0 - \epsilon, m_0 + \epsilon)$ let z_m be the largest of all possible maximum. So there exist $\delta(> 0)$ such that $g(z_m + t) + g(z_m - t) - 2g(z_m) \leq 0$ for all $t \in (0, \delta)$. Hence for all $s > 0$,

$$s^3 \int_0^\delta K(s,t) \left[\frac{g(z_m + t) + g(z_m - t) - 2g(z_m)}{2} \right] dt \leq 0.$$

Therefore $\overline{SKD}_2 g(z_m) \leq 0$ and so $\overline{SKD}_2 f(z_m) \leq 0$. Hence $z_m \in N$. Since $f(x)$ is K-smooth of order two at z_m so is $g(x)$ and $g(x)$ is maximum at z_m, so by Lemma $\overline{SKD}_2 g(z_m) = 0$. This implies $\overline{SKD}_2 f(z_m) = -m$. Thus for each choice of $m \in (m_0 - \epsilon, m_0 + \epsilon)$ we have $\overline{SKD}_2 f(z_m) = -m$. This establish a one-one correspondence between each $m \in (m_0 - \epsilon, m_0 + \epsilon)$ and $z_m \in N$, which contradicts the countability of N. So f is convex on (a,b).

Corollary 3.9. *Let $f : (a,b) \to \boldsymbol{R}$ be such that*
(i) f is continuous on (a,b).
(ii) $\overline{SKD}_2 f(x) = 0$ for all $x \in (a,b)/N$ where N is a countable susbset of (a,b).
(iii) f is smooth of order two on (a,b) w. r. t same kernel.
Then f is linear on (a,b).

Theorem 3.10. *Let $f : (a,b) \to \boldsymbol{R}$ be continuous on (a,b) and $\overline{SKD}_1 f(x) \geq 0$ for all $x \in (a,b)/N$ where N is a countable subset of (a,b) and kernel is $K(s,t)$ then f is non-decreasing on (a,b).*

Proof: Let $c \in (a,b)$ and let $F(x) = \int_c^x f(x)dx$. Therefore $F'(x)$ exists on (a,b) and $F'(x) = f(x)$ for all $x \in (a,b)$.

So for each $s > 0$ and for fixed $\delta > 0$ with $a < x - \delta < x + \delta < b$ the integral

$$s^3 \int_0^\delta K(s,t) \left[\frac{F(x+t) + F(x-t) - 2F(x)}{2} \right] dt$$

$$= s^2 \left[\frac{F(x+t) + F(x-t) - 2F(x)}{2} \right] \frac{K(s,\delta)}{-1} + s^2 \int_0^\delta K(s,t) \left[\frac{f(x+t) - f(x-t)}{2} \right] dt$$

Taking lim sup as $s \to \infty$ we get $\overline{SKD}_2 F(x) = \overline{SKD}_1 f(x)$.

Hence by Theorem 3. 8, the function $F(x)$ is convex so $F'(x) = f(x)$ is non- decreasing on (a,b).

Corollary 3.11. *Let $f : (a,b) \to \boldsymbol{R}$ be continuous on (a,b) and $\overline{SKD}_1 f(x) = 0$ for all $x \in (a,b)/N$ where N is a countable subset of (a,b) then f is constant on (a,b).*

Corollary 3.12. *Let $f : [a,b] \to \boldsymbol{R}$ be continuous on (a,b) and $\overline{SKD}_1 f(x)$ exists in (a,b) then there are points ξ_1 and ξ_2 in (a,b) such that*

$$\overline{SKD}_1 f(\xi_1) \leq \tfrac{f(b) - f(a)}{b-a} \leq \overline{SKD}_1 f(\xi_2).$$

Proof: Let $k = \frac{f(b)-f(a)}{b-a}$.

If possible suppose that there does not exist any $\xi \in (a,b)$ such that $\overline{SKD}_1 f(\xi_1) \leq k$. Let $g(x) = f(x) - k(x-a)$. Then $\overline{SKD}_1 g(x)$ exists in (a,b) an $\overline{SKD}_1 g(x) > 0$ in (a,b). Therefore $g(x)$ is increasing in $[a,b]$. So $g(b) > g(a)$ and $\overline{SKD}_1 f(\xi_1) > k$ which contradicts our assumption. So there exists ξ_1 in (a,b) such that

$$\overline{SKD}_1 f(\xi_1) \leq \frac{f(b)-f(a)}{b-a}.$$

Similarly it can be proved that there exists ξ_2 in (a,b) such that

$$\frac{f(b)-f(a)}{b-a} \leq \overline{SKD}_1 f(\xi_2).$$

Therefore our result is proved.

Lemma 3.13. *If $f^{(n-2)}$ and $SKD_n f$ exist at $x \in (a,b)$ then for all $i = 2,3,...,n-1, SKD_i(f^{(n-i)})$ exist at $x \in (a,b)$ and equals to $SKD_n f(x)$.*

Proof: The proof is shown for odd n and the proof of even n will be similar.

Let $n = 2m+1$ and for a $\delta > 0$ let us define

$$I_{2m+1}(f,x) = s^{2m+2} \int_0^\delta K(s,t) \left[\frac{f(x+t)+f(x-t)}{2} - \sum_{i=0}^{m-1} \frac{t^{2i+1}}{(2i+1)!} f^{(2i+1)}(x) \right] dt.$$

Integrating by parts we obtain

$$I_{2m+1}(f,x) = -s^{2m+1} K(s,\delta) \left[\frac{f(x+\delta)+f(x-\delta)}{2} - \sum_{i=0}^{m-1} \frac{\delta^{2i+1}}{(2i+1)!} f^{(2i+1)}(x) \right] +$$
$$s^{2m+1} \int_0^\delta K(s,t) \left[\frac{f'(x+t)+f'(x-t)}{2} - \sum_{i=0}^{m-1} \frac{t^{2i+1}}{(2i+1)!} (f')^{(2i)}(x) \right] dt.$$

Therefore, $SKD_{2m+1} f(x) = \lim_{s\to} I_{2m+1}(f,x) = I_{2m}(f',x)$.

The process after $(2m-k+1)$ time $(k>1)$ repetition gives

$$SKD_{2m+1} f(x) = \lim_{s\to} I_k(f^{(2m-k+1)},x) = SKD_k(f^{(2m-k+1)})(x).$$

This completes the proof .

Lemma 3.14. *If $f^{(n-2)}$ exists at $x \in (a,b)$ and f is K-smooth w. r. t a kernel $K(s,t)$ of order n at x for all $k = 2,3,...,n-1$, $f^{(n-k)}$ is K-smooth w. r. t $K(s,t)$ of order k at x.*

It's proof is similar to the proof of Lemma 3. 13.

Theorem 3.15. *Let $f : [a,b] \to \boldsymbol{R}$ be a function such that*

(i) $f^{(n-2)}$ exists in (a,b),

(ii) $\overline{SKD}_2 f(x) \geq 0$ for all $x \in (a,b) - E$ where E is a countable subset of (a,b).

(iii) f is smooth of order n on E w. r. t same kernel.

Then $f^{(n-2)}$ is convex on (a,b).

Proof: From Lemma 3. 13 it can be observed that $\overline{SKD}_n f(x) = \overline{SKD}_2 f^{(n-2)}(x)$ for $x \in (a,b)$. Also by Lemma 3. 14, $f^{(n-2)}$ is smooth of order two at each x in E. Hence by Theorem , $f^{(n-2)}$ is convex on (a,b).

Lemma 3.16. *Let f be continuous in (a,b), $SKD_1 f(x)$ exists and is continuous on (a,b) then f' exists on (a,b) with equal value.*

Proof: Let

$$F(x) = \int_a^x SKD_1 f(t)dt.$$

Thus $F'(x) = SKD_1 f(x)$ for all $x \in (a,b)$.
Suppose $\phi = f - F$, then $SKD_1\phi(x) = SKD_1 f(x) - F'(x) = 0$ for all $x \in (a,b)$.
By Corollary 3.11, we get that ϕ is constant. Since $F'(x)$ exists in (a,b), $f'(x)$ exists on (a,b) and $f'(x) = F'(x) = SKD_1 f(x)$ for all $x \in (a,b)$.

Theorem 3.17. *Let $f : [a,b] \to \boldsymbol{R}$ is continuous and let $SKD_{2k-1} f(x)$ exists for $k = 1,2,...,m$ and are continuous on (a,b). Then $f^{(2m-1)}$ exists and $SKD_{2m-1} f = f^{(2m-1)}$ in (a,b).*

Proof: From Lemma 3. 16 it can be concluded that the theorem is true for $m = 1$. Let it be true for $m = n$, so $f^{(2n-1)}$ and $f^{(2n-1)}(x) = SKD_{2n-1} f(x)$. We prove it for $m = n+1$. Let

$$F(x) = \int_a^x dt \int_a^t SKD_{2n+1} f(x)dx.$$

Since $SKD_{2n+1} f(x)$ is continuous in (a,b), $F''(x) = SKD_{2n+1} f(x)$.
Let $\sigma = f^{(2n-1)} - F$. So by lemma 3. 13, $SKD_2\sigma = SKD_2(f^{(2n-1)}) - F'' = 0$.
Therefore σ is linear and so $f^{(2n+1)}$ exists and $F''(x) = f^{(2n+1)}(x)$ for all x in (a,b). Hence $SKD_{2n+1} f(x) = f^{(2n+1)}(x)$ in (a,b), which shows that the theorem is true for $m = n+1$. By Induction the theorem is proved.

Theorem 3.18. *Let $f : [a,b] \to \boldsymbol{R}$ is continuous and let $SKD_{2k} f(x)$ exists for $k = 1,2,...,m$ and are continuous on (a,b). Then $f^{(2m)}$ exists and $SKD_{2m} f = f^{(2m)}$ in (a,b).*

The proof is similar to the previous proof of odd case.

4 Baire 1 and Baire * 1 property of the derivative

Lemma 4.1. *Let $f : (a,b) \to \boldsymbol{R}$ be continuous and let $E \subset (a,b)$. Let $SKD_{2i} f(x)$ exist finitely on E and let $SKD_{2i} f(x)/E$ is continuous for $i = 1,2,...,n$. Let $c \in E$ be a limit point of E. Let $(c-3\delta, c+3\delta) \subset (a,b)$. If*

$$F(x) = \int_0^\delta K(s,t)\left[\frac{f(x+t)+f(x-t)}{2} - \sum_{i=0}^{n} \frac{t^{2i}}{(2i)!} SKD_{2i} f(x)\right] dt,$$

for $x \in (c-3\delta, c+3\delta) \bigcap E$. Then $\lim\limits_{x \to c, x \in E} = F(c)$.

Proof: Let $\psi(x,t) = K(s,t)\left[\frac{f(x+t)+f(x-t)}{2} - \sum_{i=0}^{n} \frac{t^{2i}}{(2i)!} SKD_{2i}f(x)\right]$.
It is easy to verify that $\psi(x,t)$ is continuous on $E \times [0,\delta]$. Hence $F(x)/E$ is continuous. Therefore,

$$\lim_{x \to c, x \in E} = F(c).$$

Theorem 4.2. *Let $f : (a,b) \to \boldsymbol{R}$ be continuous and let $Q \subset E \subset (a,b)$. Let $SKD_{n-2}f(x)$ exists finitely on E and let $SKD_n f(x) > \infty$ and $SKD_n f(x) < \infty$ for $x \in Q$. Then there exists a sequence Q_k such that $Q \subset Q_k \subset E, Q_k$ is closed relative to E, for each k and $SKD_i f(x)/Q_k$ is continuous for $i = n-2, n-4, ...,$ and $k = 1, 2, ...$.*

Proof: We prove the result for $n = 3$ and then use mathematical induction to prove the theorem. Let $\delta > 0$ be fixed number. For each positive integer k define
$E_k = \{x \in E : s^4 \int_0^\delta K(s,t)\left[\frac{f(x+t)-f(x-t)}{2} - tSKD_1 f(x)\right] dt > k$ and
$s^4 \int_0^\delta K(s,t)\left[\frac{f(x+t)-f(x-t)}{2} - tSKD_1 f(x)\right] dt < k$ for $s > k\}$.
Let $Q_k = \overline{E_k}$ where $\overline{E_k}$ is the closure of E_k relative to E. Then $Q \subset \cup_{k=1}^{\infty} Q_k \subset E$. We are to prove that $SKD_1 f(x)/Q_k$ is continuous for $k = 1, 2, ...$ Let $c \in Q_k$ be such that c is a limit point of Q_k. Let $x \in E_k$ then

$$s^4 \int_0^\delta K(s,t)\left[\frac{f(x+t)-f(x-t)}{2} - tSKD_1 f(x)\right] dt > k \text{ for } s > k.$$

So we have for $s > k$

$$s^4 \int_0^\delta K(s,t)\left[\frac{f(x+t)-f(x-t)}{2}\right] dt > s^4 \int_0^\delta K(s,t)\left[tSKD_1 f(x)\right] dt - k \text{ for } s > k.$$
$$= \left[s^2 + o(1)\right] SKD_1 f(x) k.$$

Dividing both side by s^2 we get

$$s^2 \int_0^\delta K(s,t)\left[\frac{f(x+t)-f(x-t)}{2}\right] dt > SKD_1 f(x)\left[1 + o(\frac{1}{s^2})\right]\frac{k}{s^2} \text{ for } s > k.$$

Letting $x \to c$ through E_k we get since f is continuous and $o(\frac{1}{s^2})$ is independent of x,

$$s^2 \int_0^\delta K(s,t)\left[\frac{f(c+t)-f(c-t)}{2})\right] dt \geq \limsup_{x \to c, x \in E_k} SKD_1 f(x)\left[1 + o(\frac{1}{s^2})\right]\frac{k}{s^2} \text{ for } s > k.$$

Letting $s \to \infty$ we get

$$SKD_1 f(c) \geq \limsup_{x \to c, x \in E_k} SKD_1 f(x). \; . \; . \; . \text{ (A)}$$

Again for $x \in E_k$,

$$s^4 \int_0^\delta K(s,t)\left[\frac{f(x+t)-f(x-t)}{2} - tSKD_1 f(x)\right] dt < k \text{ for } s > k.$$

Therefore,

$$s^4 \int_0^\delta K(s,t)\left[\frac{f(x+t)-f(x-t)}{2}\right] dt < s^4 \int_0^\delta K(s,t)\left[tSKD_1 f(x)\right] dt + k$$
$$= \left[s^2 + o(1)\right] SKD_1 f(x) + k.$$

Dividing both side by s^2 we get

$$s^2\int_0^\delta K(s,t)\left[\frac{f(x+t)-f(x-t)}{2}\right]dt<\left[1+o(\frac{1}{s^2})\right]SKD_1f(x)+\frac{k}{s^2}.$$

Letting $x\to c$ through E_k we get since f is continuous

$$s^2\int_0^\delta K(s,t)\left[\frac{f(c+t)-f(c-t)}{2}\right]dt\le\liminf_{x\to c,x\in E_k}SKD_1f(x)+\frac{k}{s^2}$$

Letting $s\to\infty$ we get

$$SKD_1f(c)\le\liminf_{x\to c,x\in E_k}SKD_1f(x).\;.\;.\;(B).$$

From (A) and (B) we get

$$SKD_1f(c)\doteq\lim_{x\to c,x\in E_k}SKD_1f(x).$$

If possible suppose that

$$SKD_1f(c)=\liminf_{x\to c,x\in E_k}<SKD_1f(c).\;.\;.\;.\;(C)$$

Choose l such that $\liminf_{x\to c,x\in E_k}<l<SKD_1f(c)$. Therefore there is a neighbourhood G of c in which there exists no point $x_0\in E_k$ such that $SKD_1f(x_0)<l$ though there is $x_1\in Q_k\bigcap G$ satisfying $SKD_1f(x_1)<l$. Obviously x_1 is a limit point of Q_k. So from a relation similar to (B) replacing c by x_1 we get

$$l>SKD_1f(x_1)=\liminf_{x\to x_1,x\in E_k}SKD_1f(x),$$

implies that there are points $x_0\in E_k\bigcap G$ such that $SKD_1f(x_0)<l$, which is a contradiction. Therefore (C) is not possible and thus

$$\liminf_{x\to c,x\in Q_k}SKD_1f(x)\ge SKD_1f(c).$$

Analogously it can be proved that,

$$\limsup_{x\to c,x\in Q_k}SKD_1f(x)\le SKD_1f(c).$$

So,

$$\lim_{x\to c,x\in Q_k}SKD_1f(x)=SKD_1f(c)..$$

Since $Q\subset Q_k\subset E$ the theorem is true for $n=3$. Now suppose that $n>3$ and n is odd. Let the theorem is true for $n=2m+1$. We prove the theorem for $n=2m+3$. Let the hypothesis of the theorem is hold for $n=2m+3$. Then $SKD_{2m+1}f(x)$ exist finitely for $x\in E$ and $SKD_{2m+3}f(x)>\infty$ and $SKD_{2m+3}<\infty$ for $x\in Q$. Since the theorem is true for $n=2m+1$, there is a sequence of sets P_r such that Pr is closed relative to E for each $r,Q\subset P_r\subset E$ and $SKD_{2i+1}f(x)/Pr$ is continuous for $i=0,1,2,...,m-1$ and $r=1,2,......$. For each pair of positive integers k,r let

$$E_{kr}=\{x\in Q\cap P_r:s^{2m+4}\int_0^\delta K(s,t)\left[\frac{f(x+t)-f(x-t)}{2}\sum_{i=0}^{n}\frac{t^{2i+1}}{(2i+1)!}SKD_{2i+1}f(x)\right]dt>-k$$

and $s^{2m+4}\int_0^\delta K(s,t)\left[\frac{f(x+t)-f(x-t)}{2}-\sum_{i=0}^{n}\frac{t^{2i+1}}{(2i+1)!}SKD_{2i+1}f(x)\right]dt<k$ for $s>k\}$ . . . (D)

Let $T_{kr}=\overline{E}_{kr}$, where E_{kr} is the closure of E_{kr} relative to E. Clearly $Q\subset\bigcup_k\bigcup_r T_{kr}\subset E$. We show that $SKD_{2m+1}f(x)/T_{kr}$ is continuous in each T_{kr}. Let $\xi\in T_{kr}$ be such that ξ is a limit point of T_{kr}. Let $x\in T_{kr}$ then by (D)

$s^{2m+4}\int_0^\delta K(s,t)\left[\frac{f(x+t)-f(x-t)}{2}-\sum_{i=0}^{n}\frac{t^{2i+1}}{(2i+1)!}SKD_{2i+1}f(x)\right]dt>k$ for $s>k$.

So , for $s>k$ we have

$s^{2m+4}\int_0^\delta K(s,t)\left[\frac{f(x+t)-f(x-t)}{2}-\sum_{i=0}^{m-1}\frac{t^{2i+1}}{(2i+1)!}SKD_{2i+1}f(x)\right]dt>s^{2m+4}\int_0^\delta K(s,t)\frac{t^{2m+1}}{(2m+1)!}SKD_{2m+1}f(x)dt$
$=SKD_{2m+1}f(x)\left[s^2+o(1)\right]-k$.

Dividing both side by s^2 we get,

$$s^{2m+2}\int_0^\delta K(s,t)\left[\frac{f(x+t)-f(x-t)}{2}-\sum_{i=0}^{m-1}\frac{t^{2i+1}}{(2i+1)!}SKD_{2i+1}f(x)\right]dt$$
$$>SKD_{2m+1}f(x)\left[1+o(\tfrac{1}{s^2})\right]-\tfrac{k}{s^2}\ .$$

for $s>k$. Since f is continuous and $SKD_{2m+1}f(x)/P_r$ is continuous for $i=0,1,...,m-1$, letting $x\to\xi$ through E_{kr} we have ,

$$s^{2m+2}\int_0^\delta K(s,t)\left[\frac{f(\xi+t)-f(\xi-t)}{2}-\sum_{i=0}^{m-1}\frac{t^{2i+1}}{(2i+1)!}SKD_{2i+1}f(\xi)\right]dt$$
$$\geq\limsup_{x\to\xi,x\in E_{kr}}SKD_{2m+1}f(x)\left[1+o(\frac{1}{s^2})\right]-\frac{k}{s^2}$$

for $s>k$. Letting $s\to\infty$ We have

$$SKD_{2m+1}f(\xi)\geq\limsup_{x\to\xi,x\in E_{kr}}SKD_{2m+1}f(x)\ .\ .\ .\ .\ (E)$$

Again for $x\in E_{kr}$,

$$s^{2m+4}\int_0^\delta K(s,t)\left[\frac{f(x+t)-f(x-t)}{2}-\sum_{i=0}^{m}\frac{t^{2i+1}}{(2i+1)!}SKD_{2i+1}f(x)\right]dt<k$$

for $s>k$. Then proceeding in a similar manner as above we get

$$SKD_{2m+1}f(\xi)\leq\liminf_{x\to\xi,x\in E_{kr}}SKD_{2m+1}f(x)\ .\ .\ .\ .\ (F)$$

From (E) and (F) we get

$$SKD_{2m+1}f(\xi)=\lim_{x\to\xi,x\in E_{kr}}SKD_{2m+1}f(x).$$

Approching as in case for $n=3$ it can be shown that

$$SKD_{2m+1}f(\xi)=\lim_{x\to\xi,x\in T_{kr}}SKD_{2m+1}f(x).\ .\ .\ (G).$$

Now we can write $\bigcup_k\bigcup_r T_{kr}$ as $\bigcup_{\lambda=1}^{r}Q_\lambda$ and from (G) it follows that $SKD_{2m+1}f(x)/Q$ is continuous for $=1,2,....$ Also Q$\subset\bigcup_k\bigcup_r T_{kr}=\bigcup Q_\lambda\subset E$ and since each T_{kr} is closed relative to E so each Q is closed relative to E. Thus the theorem is true for $n = 2m+3$. This completes the proof for odd case.

To prove the theorem for even n, first the theorem is proved for $n = 4$ in the analogous way and induction is used as previous. The detail is avoided.

Corollary 4.3. *Under the hypothesis of Theorem 4. 2, if* $E = Q$ *then* $SKD_k f(x) \in B_1(E)$ *for* $k = n-2, n-4, ...$

Theorem 4.4. *Let* $f : (a,b) \to \boldsymbol{R}$ *be continuous and* $E \subset (a,b)$ *be closed. If* $SKD_n f(x)$ *exists finitely on* E *then* $SKD_n f(x) \in B_1(E)$.

Proof: The derivatives $SKD_1 f(x)$ and $SKD_2 f(x)$ if exist on E then obviously Baire 1. So we take $n > 2$. Since $SKD_n f(x)$ exists finitely on E so by previous Theorem there exists a sequence of sets Q_k such that each Q_k is closed relative to $E, \bigcup Qk = E$ and $\text{SKD}_i f(x)/Q_k$ is continuous for $i = n-2, n-4, ..., k = 1, 2, ...$. Let δ be a fixed number such that $0 < \delta < min\{\inf E - a, b - \sup E\}$. For each positive integer m define

$$f_m(x) = m^{n+1} \int_0^\delta K(m,t) \left[\frac{f(x+t)+f(x-t)}{2} \sum_{r=0}^{\frac{n-2}{2}} \frac{t^{2r}}{(2r)!} SKD_{2r} f(x) \right] dt, \text{ when } x \in E \text{ and } n \text{ is even.}$$

$$= m^{n+1} \int_0^\delta K(m,t) \left[\frac{f(x+t)-f(x-t)}{2} \sum_{r=0}^{\frac{n-2}{2}} \frac{t^{2r+1}}{(2r+1)!} SKD_{2r+1} f(x) \right] dt,$$

when $x \in E$ and n is odd.

Then, $f_m(x)/Q_k$ is continuous for each m and for each k . Also $\lim_{m\to\infty} f_m(x) = SKD_n f(x)$ for $x \in E$. Let $F \subset E$ be a closed set . Then $F = \bigcup_{k=1}^{\infty} (Q_k \bigcap F)$. By Baires Theorem there is (u,v) and a k_0 such that $\Phi \neq F \bigcap (u,v) \subset (Q_{k_0} \bigcap F) \bigcap (u,v)$.

Since f_m/Q_{k_0} is continuous, $SKD_n f(x) \in B_1(F \bigcap (u,v))$. Hence $SKD_n f(x) \in B_1(E)$.

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