NEW MEANS AND INEQUALITIES

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Abstract. In this paper, based on proportions, three new means are constructed. Some characteristic properties of these new means are verified and new inequalities involving them have been established.

1. Introduction

For $e, m, f > 0$, based on proportions the means $M(e, f)$ are defined. The following table 1.1 shows 10 Greek means of which 4 are un-named [12].

Sl.No.	Proportion	Name of Mean	Notation
1	$\frac{e-m}{m-f} = \frac{e}{e}$	Arithmetic Mean	$A = \frac{e+f}{2}$
2	$\frac{e-m}{m-f} = \frac{m}{f} = \frac{e}{m}$	Geometric Mean	$G = \sqrt{ef}$
3	$\frac{e-m}{m-f} = \frac{e}{f}$	Harmonic Mean	$H = \frac{2ef}{e+f}$
4	$\frac{e-m}{m-f} = \frac{e}{f}$	Contra harmonic Mean	$C = \frac{e^2+f^2}{e+f}$
5	$\frac{e-m}{m-f} = \frac{f}{m}$	First kind of Contra geometric Mean	$F_5 = \frac{e-f+\sqrt{(e-f)^2+4f^2}}{2}$
6	$\frac{e-m}{m-f} = \frac{m}{e}$	second kind of contra geometric Mean	$F_6 = \frac{f-e+\sqrt{(e-f)^2+4e^2}}{2}$
7	$\frac{e-m}{e-f} = \frac{f}{e}$	un named Mean	$F_7 = \frac{e^2-ef+f^2}{e}$
8	$\frac{e-m}{e-f} = \frac{m}{e}$	un named Mean	$F_8 = \frac{e^2}{2e-f}$
9	$\frac{m-f}{e-f} = \frac{f}{e}$	un named Mean	$F_9 = \frac{f(2e-f)}{e}$
10	$\frac{m-f}{e-f} = \frac{f}{m}$	un named Mean	$F_{10} = \frac{f+\sqrt{f(4e-3f)}}{2}$

Table 1.1

The construction of new means are studied in [1, 2, 4, 5, 6, 7] inequalities involving well known means are discussed in [3, 8, 9, 10, 11, 13].

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2. Structure of new mean

In this section, three new means are defined and clearly observed that, the means $F_i \neq F_j$, where $i \neq j$ and $1 \leq i, j \leq 10$.

Definition 2.1. *For $e > m > f > 0$, choose $\beta = \frac{e}{f}$ or $\frac{f}{m}$ or $\frac{m}{e}$, and $\beta \in (0,1)$.*

In the proportion 9 of the table 1.1, (ie. serial number 9), replace e by $M(e,f)$ and f by $N(e,f)$, then

$$\beta = \frac{m - N(e,f)}{M(e,f) - N(e,f)}, \tag{2.1}$$

where $M(e,f)$ and $N(e,f)$ are any two distinct means of above 10 Greek means (i.e., F_i, where $1 \leq i \leq 10$).
On simplifying and replacing m by R leads to

$$R = R(e,f;\beta) = \beta M(e,f) + (1-\beta)N(e,f) \tag{2.2}$$

The proportion 8 of the table 1.1, (ie. serial number 8), replace e by $M(e,f)$ and f by $N(e,f)$, then

$$\beta = \frac{M(e,f) - m}{M(e,f) - N(e,f)} \tag{2.3}$$

On simplifying and replacing m by S,

$$S = S(e,f;\beta) = \beta N(e,f) + (1-\beta)M(e,f) \tag{2.4}$$

In the proportion 6 of the table 1.1, (ie. serial number 6), replace e by $M(e,f)$ and f by $N(e,f)$, then

$$\beta = \frac{M(e,f) - m}{m - N(e,f)} \tag{2.5}$$

On simplifying and replacing m by T leads to

$$T = T(e,f;\beta) = \frac{1}{1+\beta}M(e,f) + \frac{\beta}{1+\beta}N(e,f) \tag{2.6}$$

Here, $R(e,f;\beta)$, $S(e,f;\beta)$, and $T(e,f;\beta)$ are the three new means.
In the proportions 2.1, 2.3 and 2.5, replace e by $ln\ e$, f by $ln\ f$, m by $ln\ m$, $M(e,f)$ by $ln\ M(e,f)$ and $N(e,f)$ by $ln\ N(e,f)$, by simplifying and replacing m by $R^{(d)}$, m by $S^{(d)}$ and m by $T^{(d)}$, the following equations are obtained.

$$R^{(d)} = R^{(d)}(e,f;\beta) = M(e,f)^{\beta} N(e,f)^{1-\beta} \tag{2.7}$$

$$S^{(d)} = S^{(d)}(e,f;\beta) = N(e,f)^{\beta} M(e,f)^{1-\beta} \tag{2.8}$$

and

$$T^{(d)} = T^{(d)}(e,f;\beta) = M(e,f)^{\frac{1}{1+\beta}} N(e,f)^{\frac{\beta}{1+\beta}} \tag{2.9}$$

Then, $R^{(d)}(e,f;\beta)$, $S^{(d)}(e,f;\beta)$ and $T^{(d)}(e,f;\beta)$ are respectively called the dual forms of the three new means $R(e,f;\beta)$, $S(e,f;\beta)$ and $T(e,f;\beta)$.

3. Some properties among new means

For any Greek mean $M(e,f) \neq N(e,f)$, the following properties are obtained by giving particular values to β

(1) $R(e,f;\frac{1}{2}) = S(e,f;\frac{1}{2})$
(2) $R(e,f;0) = S(e,f;1) = N(e,f)$
(3) $R(e,f;1) = S(e,f;0) = T(e,f;0) = M(e,f)$
(4) $T(e,f;1) = \frac{1}{2}[S(e,f;0) + S(e,f;1)] = \frac{1}{2}(M+N)$
(5) $T(e,f;1) = \frac{1}{2}[R(e,f;0) + R(e,f;1)] = \frac{1}{2}(M+N)$
(6) $T(e,f;1) = \frac{1}{2}[S(e,f;0) + R(e,f;0)] = \frac{1}{2}(M+N)$
(7) $T(e,f;1) = \frac{1}{2}[S(e,f;1) + R(e,f;1)] = \frac{1}{2}(M+N)$
(8) $R(e,f;1) \leq T(e,f;1) \leq S(e,f;1)$
(9) $S(e,f;0) \leq T(e,f;0) \leq R(e,f;0)$

In particular, take $N(e,f) = A(e,f) = A$ and $M(e,f) = H(e,f) = H$, then the following deductions and inequalities are obtained.

(1) $R(e,f;\frac{1}{2}) = S(e,f;\frac{1}{2}) = \frac{A(e,f)+H(e,f)}{2} = C(e,f)$
(2) $R(e,f;0) = S(e,f;1) = A(e,f)$
(3) $R(e,f;1) = S(e,f;0) = T(e,f;0) = H(e,f)$
(4) $T(e,f;1) = \frac{1}{2}[S(e,f;0) + S(e,f;1)] = \frac{A(e,f)+H(e,f)}{2} = C(e,f)$
(5) $T(e,f;1) = \frac{1}{2}[R(e,f;0) + R(e,f;1)] = \frac{A(e,f)+H(e,f)}{2} = C(e,f)$
(6) $T(e,f;1) = \frac{1}{2}[S(e,f;0) + R(e,f;0)] = \frac{A(e,f)+H(e,f)}{2} = C(e,f)$
(7) $T(e,f;1) = \frac{1}{2}[S(e,f;1) + R(e,f;1)] = \frac{A(e,f)+H(e,f)}{2} = C(e,f)$
(8) $R(e,f;1) \leq T(e,f;1) \leq S(e,f;1)$
(9) $S(e,f;0) \leq T(e,f;0) \leq R(e,f;0)$

4. Some inequalities among new means

The inequalities involving the new means $R(e,f;\beta)$, $S(e,f;\beta)$ and $T(e,f;\beta)$ and the validity of some double inequalities are presented in the form of theorems.

Theorem 4.1. *For $e,f > 0$ and $M(e,f) \leq N(e,f)$, then with usual notations the following inequalities are holds good;*

(1) $R(e,f;\beta) < T(e,f;\beta) < S(e,f;\beta)$ *for* $\beta \in (0.62, 1)$
(2) $T(e,f;\beta) < R(e,f;\beta) < S(e,f;\beta)$ *for* $\beta \in (0, 0.5)$

Proof: Consider $S - R = [N\beta + (1-\beta)M] - [M\beta + (1-\beta)N]$
$= (1-2\beta)M - (1-2\beta)N$
$= (1-2\beta)(M-N) > 0$, if $\beta > \frac{1}{2}$
$= (1-2\beta)(M-N) < 0$, if $\beta < \frac{1}{2}$
That is

$$S > R \quad if\ \frac{1}{2} < \beta < 1 \tag{4.1}$$

$$S < R \quad if\ 0 < \beta < \frac{1}{2} \tag{4.2}$$

Similarly by considering $S - T = [N\beta + M(1-\beta)] - \left[\left(\frac{\beta}{1+\beta}\right)N + \left(\frac{1}{1+\beta}\right)M\right]$
$= \left[\frac{\beta+\beta^2-\beta}{1+\beta}\right]N + \left[\frac{(1-\beta^2-1)}{1+\beta}\right]M$
$= \left[\frac{\beta^2}{1+\beta}\right](N-M) > 0$
That is

$$S > T \quad for\ all\ \beta \in (0,1) \tag{4.3}$$

and $R - T = [\beta M + (1-\beta)N] - \left[\left(\frac{\beta}{1+\beta}\right)N + \left(\frac{1}{1+\beta}\right)M\right]$
$= \left(\frac{\beta+\beta^2-1}{1+\beta}\right)M + \left(\frac{1-\beta^2-\beta}{1+\beta}\right)N$
$= \left(\frac{\beta^2+\beta-1}{1+\beta}\right)(M-N) > 0$
which is equivalently written as;

$$R - T = \frac{\left(\beta + \frac{\sqrt{5}+1}{2}\right)\left(\beta + \frac{1-\sqrt{5}}{2}\right)}{1+\beta}(M-N) \tag{4.4}$$

$$R - T > 0 \quad if\ \beta + \frac{1-\sqrt{5}}{2} < 0\ or\ \beta < 0.61 \tag{4.5}$$

$$R - T < 0 \quad if\ \beta + \frac{1-\sqrt{5}}{2} > 0\ or\ \beta > 0.62 \tag{4.6}$$

By combining the eqs $(6.4.1) - (6.4.6)$ leading to,
$R < T < S, \qquad \beta \in (0.62, 1)$ and $\quad T < S < R, \qquad \beta \in (0, \frac{1}{2})$
To substantiate the result obtained, fix particular values for the variables.
Take $M = \sqrt{2}$, $N = 1.5$, since $\beta \in (0.62, 1) = 0.64$
$R = M\beta + (1-\beta)N = 1.444$
$T = \left[\left(\frac{\beta}{1+\beta}\right)N + \left(\frac{1}{1+\beta}\right)M\right] = 1.447$
$S = N\beta + M(1-\beta) = 1.4690$
Hence
$R < T < S, \qquad \beta \in (0.62, 1)$
Similarly for $M = \sqrt{2}$, $N = 1.5$, since $\beta \in (0, 0.5) = 0.3$
$R = M\beta + (1-\beta)N = 1.474$
$T = \left[\left(\frac{\beta}{1+\beta}\right)N + \left(\frac{1}{1+\beta}\right)M\right] = 1.433$
$S = N\beta + M(1-\beta) = 1.439$
Therefore
$T < S < R \qquad \beta \in (0, \frac{1}{2})$
This complete the proof of the Theorem (6.4.1).

As an application the following inequalities are established for real value of $y \in (0,1)$.

Theorem 4.2. *For $y \in (0,1)$ then with usual notations the following inequalities are holds good;*

(1) $1 + y \geq 2^y$
(2) $2 - y \geq 2^{1-y}$
(3) $1 + \frac{1}{1+y} \geq 2^{\frac{1}{1+y}}$
(4) $1 - \frac{y}{2} \geq \frac{1}{2^y} \geq \frac{1}{1+y}$
(5) $\frac{1}{2^y} \geq \frac{1}{1+y} \geq 2^{\left(\frac{1}{1+y}\right)} - 1$

Proof: By setting $M(e, f) = M = 2$, $N(e, f) = N = 1$ and $\beta \in (0, 1)$, then new means and their dual forms are given by;

(1) $R = 2\beta + (1 - \beta) = \beta{+}1$
(2) $R^d = 2^\beta$
(3) $S = 2(1 - \beta) + \beta = 2 - \beta$
(4) $S^d = 2^{1-\beta}$
(5) $T = \frac{2+\beta}{1+\beta}$
(6) $T^d = 2^{\frac{1}{1+\beta}}$

From Arithmetic Geometric mean inequality and the results proved in theorem 4.1 leads to the following, which are holds for all $\beta \in (0, 1)$.

(1) $R = 2\beta + (1 - \beta) = 1 + \beta \geq R^d = 2^\beta$
(2) $S = \beta + 2(1 - \beta) = 2 - \beta \geq S^d = 2^{1-\beta}$
(3) $T = \frac{2+\beta}{1+\beta} \geq T^d = 2^{\frac{1}{1+\beta}}$

Replace β by y and $y \in (0, 1)$ then

$$1 + y \geq 2^y \tag{4.7}$$

$$2 - y \geq 2^{1-y} \tag{4.8}$$

$$1 + \frac{1}{1+y} \geq 2^{\left(\frac{1}{1+y}\right)} \tag{4.9}$$

By combining eqs 4.7 and 4.8 gives the inequality

$$1 - \frac{y}{2} \geq \frac{1}{2^y} \geq \frac{1}{1+y} \tag{4.10}$$

By combining eqs 4.7 and 4.9 gives the inequality

$$\frac{1}{2^y} \geq \frac{1}{1+y} \geq 2^{\left(\frac{1}{1+y}\right)} - 1 \tag{4.11}$$

To substantiate the result obtained, fix particular values for the variables. since $y \in (0, 1)$, fix $y = 0.5$ for the above equations from (6.4.7) to (6.4.11)
$1 + 0.5 \geq 2^{0.5}$ implies $1.5 \geq 1.412$
$2 - y \geq 2^{1-0.5}$ implies $1.5 \geq 1.414$
$1 + \frac{1}{1+0.5} \geq 2^{\left(\frac{1}{1+0.5}\right)}$ implies $1.666 \geq 1.587$
$1 - \frac{0.5}{2} \geq \frac{1}{2^{0.5}} \geq \frac{1}{1+0.5}$ implies $0.75 \geq 0.707 \geq 0.666$
$\frac{1}{2^{0.5}} \geq \frac{1}{1+0.5} \geq 2^{\left(\frac{1}{1+0.5}\right)} - 1$ implies $0.707 \geq 0.666 \geq 0.587$

5. Conclusion

New means are constructed and a set of double inequalities are established.

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