Solving some fractional-order mathematical physics equations in time using the BONAZEBI YINDOULA Joseph method

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Abstract

In this paper, the BYJ method (a combination of the reduced differential transformation method and successive picard approximations)is used to construct the exact solution of some nonlinear fractional differential equations in time.

Keywords: Reduced Differential Transform Method,fractional partial differential equations, Caputo derivation, BYJ Method,time-fractional

1 Introduction

Partial differential equations of integer or fractional order appear in almost every field of physics, applied science and engineering [1, 2]. To better understand these physical phenomena and apply them to practical scientific research, it is important to find their exact solutions. The study of the exact solution of these equations is interesting and important. Over the last few decades, many authors have studied the solution of such equations using various methods developed. Recently, the variational iteration method (VIM) [3, 4],has been applied to treat various types of nonlinear problems, for example, fractional differential equations [5], nonlinear differential equations [5], nonlinear thermo-elasticity [6], wave equations [5]. In references [7, 8], the Adomian decomposition method (ADM), the homotopy perturbation method (HPM), the homotopy analysis method (HAM) and the parameter variation method (VPM), the reduced differential transformation method (RDTM) introduced by Kaskin and Oturanc [9, 10] are successfully applied to obtain the exact solution of integer and fractional order partial differential equations. In this article, we have used the BYJ method introduced by the author to construct a solution for some fractional-order partial differential equations. The BYJ technique is an iterative procedure for obtaining a solution in the form of a Taylor series converging to the exact solution. The method is semi-analytical for solving fractional, nonlinear, homogeneous and inhomogeneous partial differential equations. Results show that the BYJ method is accurate, efficient and requires less effort than other analytical and numerical methods.

2 BONAZEBI YINDOULA Joseph (BYJ) method

The Reduced Differential Transformation Method (RDTM) was first proposed by Turkish mathematician Yildiray Keskin in 2009. This method is applicable to a wide class of linear and nonlinear problems with approximations that converge rapidly to the exact solution if it exists.

The BYJ method, on the other hand, is based on a combination of the reduced differential transformation method (RDTM), successive approximations and Picard's principle, and is used to solve classical and fractional partial differential equations.

2.1 Preliminary

Consider a two-variable function $u(x,t)$ which is analytic and $k-fois$ continuously differentiable with respect to x and time t. Assume that it can be represented as the product of two single-variable functions, i.e. $u(x,t) = f(x).g(t)$ then using the properties of the one-dimensional differential transformation, the function can be represented as follows:

$$u(x,t) = \left(\sum_{i=0}^{\infty} F(i)x^i \sum_{j=0}^{\infty} G(j)t^j \right) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u(i,j)x^i t^j \tag{1}$$

where $u(i,j) = F(i)G(j)$is called the spectrum of $u(x,t)$.

The basic definitions and operations of the reduced differential transformation method are introduced as follows

2.1.1 Definition 1

If $u(x,t)$ is analytic and continuously differentiable with respect to the space variable x and time t, then the reduced differential transform is given by:

$$u_k(x) = \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} u(x,t) \right]_{t=t_0}, k \in \mathbb{N} \tag{2}$$

where the t-dimensional function $U_k(x)$ is a transformed function.

Lowercase $u(x;t)$represents the original function while uppercase $U_k(x)$ represents the transformed function.

The inverse reduced differential transformation of $U_k(x)$ is defined by:

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} u(x,t) \right]_{t=t_0} (t-t_0)^k = \sum_{k=0}^{\infty} u_k(x) \, (t-t_0)^k \tag{3}$$

Among the above definitions, we can find that the concept of reduced differential transformation is derived from power series .

2.1.2 Definition 2

If the function $u(x;t)$ is analytic and continuous differentiable for t, we have :

$$u_k(x) = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t) \right]_{t=t_0}, k \in \mathbb{N} \tag{4}$$

where α is a parameter describing the order of the fractional derivative in time and the t-dimensional spectral function $U_k(x)$ is the transformed function.

The inverse reduced differential transformation of $U_k(x)$ is defined by:

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t) \right]_{t=t_0} (t-t_0)^k = \sum_{k=0}^{\infty} u_k(x) \, (t-t_0)^{k\alpha} \tag{5}$$

2.1.3 Table of fundamental operations for reduced differential transformation

Original function	Reduced differential transformation function
$u(x,t)$	$U_k(x) = \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} u(x) \right]_{t=t_0}$
$w(x,t) = \frac{\partial^{n\alpha}}{\partial x^{n\alpha}} u(x,t)$ and $n \in \mathbb{N}$	$W_k(x) = \frac{\partial^{n\alpha}}{\partial x^{n\alpha}} U_k(x)$
$w(x,t) = \frac{\partial^{n\alpha}}{\partial t^{n\alpha}} u(x,t)$ and $n \in \mathbb{N}$	$W_k(x) = \frac{\Gamma(1+(k+n)\alpha)}{\Gamma(1+k\alpha)} U_{k+n}(x)$

2.1.4 The Mittag-Leffler function

The Mittag-Leffler function plays a very important role in the theory of integer-order differential equations. It is also widely used in the search for solutions of fractional-order differential equations. This function was introduced by G. M. Mittag-Leffler in 1905.

Definition

The Mittag Leffler function is the function denoted $E_\alpha(z)$ defined by :

$$E_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha+1)} \tag{6}$$

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where z is a complex number, α is a strictly positive real number

NB: The generalization of this function for two parameters is [17, 19] :

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0 \tag{7}$$

This function was introduced by R.P Agarwal and Erdelyi in 1953-1954.

2.2 Description of the BYJ method for a fractional partial differential equation

Consider the following non-linear partial differential equation of fractional time order.

$${}^{C}D_t^{n\alpha}u(x,t) + Ru(x,t) + Nu(x,t) = f(x,t), t > 0, x \in \mathbb{R} \tag{8}$$

with initial conditions

$$\frac{\partial^i u(x,0)}{\partial t^i} = g_i(x), i = 0, 1, 2, 3, \cdots, n-1 \tag{9}$$

Where ${}^{C}D_t^{n\alpha}$ denotes the fractional derivative in the Caputo sense of order n , where $n-1 < n\alpha < n, n \in \mathbb{N}^*, R$ is a linear operator, N is a non-linear operator and $f(x,t)$ is the source term.

Equation (8) can still be written as follows.

$${}^{C}D_t^{n\alpha}u(x,t) = f(x,t) - Ru(x,t) - Nu(x,t) \tag{10}$$

Applying the reduced differential transformation to (10) gives :

$$\frac{\Gamma(k\alpha + n\alpha + 1)}{\Gamma(k\alpha + 1)} U_{k+n}(x) = F_k(x) - R(U_k(x)) - N(U_k(x)), k \in \mathbb{N} \tag{11}$$

Where $U_{k+n}(x), L(U_k(x)), N(U_k(x))$ and $F_k(x)$ are the transformations of the terms ${}^{C}D_t^{n\alpha}u(x,t), Ru(x,t), Nu(x,t)$ and $f(x,t)$ respectively.Using the reduced transformations on the initial conditions (9), we obtain

$$U_0(x) = g_0(x), U_1(x) = g_1(x), \cdots, U_{n-1}(x) = g_{n-1}(x) \tag{12}$$

Applying the method of successive approximations to (11) , we get

$$\frac{\Gamma(k\alpha + n\alpha + 1)}{\Gamma(k\alpha + 1)} U_{k+n}^p(x) = F_k(x) - R(U_k^p(x)) - N\left(U_k^{p-1}(x)\right), k \in \mathbb{N}, \forall p \geq 1 \tag{13}$$

with initial conditions

$$U_0^p(x) = g_0(x), U_1^p(x) = g_1(x), \cdots, U_{n-1}^p(x) = g_{n-1}(x) \tag{14}$$

First iteration

For $p = 1$ we have :

$$\begin{cases} \frac{\Gamma(k\alpha+n\alpha+1)}{\Gamma(k\alpha+1)} U_{k+n}^{1}(x) = F_k(x) - R\left(U_k^1(x)\right) - N\left(U_k^0(x)\right), k \in \mathbb{N} \\ \\ U_0^1(x) = g_0(x), U_1^1(x) = g_1(x), \cdots, U_{n-1}^1(x) = g_{n-1}(x) \end{cases} \tag{15}$$

Applying Picard's principle, i.e. there exists $U_k^0(x)$ such that $N\left(U_k^0(x)\right) = 0$
(15) becomes

$$\begin{cases} \frac{\Gamma(k\alpha+n\alpha+1)}{\Gamma(k\alpha+1)} U_{k+n}^{1}(x) = F_k(x) - R\left(U_k^1(x)\right), k \in \mathbb{N} \\ U_0^1(x) = g_0(x), \\ U_1^1(x) = g_1(x) \\ \vdots \\ U_{n-1}^1(x) = g_{n-1}(x) \end{cases} \tag{16}$$

and by a simple iterative calculation we obtain the values $U_k^1(x)$ for $k = 0, 1, 2, 3, \cdots$

Thus the inverse fractional reduced differential transformation of the set of values $\left\{U_k^1(x)\right\}_{k=0}^{k=N}$ gives the approximate N-term solution as follows:

$$u_N^1(x,t) = \sum_{k=0}^{n} U_k^1(x) t^{k\alpha} \tag{17}$$

Consequently, the exact solution of the problem at the first iteration is given by

$$u^1(x,t) = \lim_{n \longrightarrow +\infty} u_N^1(x,t) \tag{18}$$

Second iteration

For $p = 2$ we have:

$$\begin{cases} \frac{\Gamma(k\alpha+n\alpha+1)}{\Gamma(k\alpha+1)} U_{k+n}^{2}(x) = F_k(x) - R\left(U_k^2(x)\right) - N\left(U_k^1(x)\right), k \in \mathbb{N} \\ \\ U_0^2(x) = g_0(x), U_1^2(x) = g_1(x), \cdots, U_{n-1}^2(x) = g_{n-1}(x) \end{cases} \tag{19}$$

Assume that $N\left(U_k^1(x)\right) = 0$ then (19) becomes

$$\begin{cases} \frac{\Gamma(k\alpha+n\alpha+1)}{\Gamma(k\alpha+1)} U_{k+n}^{2}(x) = F_k(x) - R\left(U_k^2(x)\right), k \in \mathbb{N} \\ \\ U_0^2(x) = g_0(x), U_1^2(x) = g_1(x), \cdots, U_{n-1}^2(x) = g_{n-1}(x) \end{cases} \tag{20}$$

which is therefore the same algorithm as in step $p = 1$

If the series $\sum_{k=0}^{+\infty} U_k^2(x)t^{k\alpha}$ is convergent, then we get

$$u^2(x,t) = \lim_{N\longrightarrow+\infty} u_N^2(x,t) \quad \text{where} \quad u_N^2(x,t) = \sum_{k=0}^{N} U_k^2(x)t^{k\alpha} \tag{21}$$

approximate solution of equation (8) in step 2

P-th iteration

Recursively, if the series $\sum_{k=0}^{+\infty} U_k^p(x)t^{k\alpha}$ is convergent for $p \geq 3$, then we get

$$u^p(x,t) = \lim_{N\longrightarrow+\infty} u_N^p(x,t) \quad \text{where } u_N^p(x,t) = \sum_{k=0}^{N} U_k^p(x)t^{k} \tag{22}$$

approximate solution of equation (8) at step p

The solution to the problem (8) is therefore

$$u(x,t) = \lim_{p\longrightarrow+\infty} u^p(x,t) \tag{23}$$

3 Applications

3.1 Example 1

Consider the following Kuramoto-Sivashinsky problem [11, 18] :

$$\begin{cases} {}^C D_t^\alpha u(x,t) = -\frac{\partial^4 u(x,t)}{\partial x^4} - 2\frac{\partial^2 u(x,t)}{\partial x^2} + u^4(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} - u^2(x,t)\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^3 \\ \\ u(x,0) = \cos x + \sin x \end{cases}, \tag{24}$$

$$0 < \alpha \leq 1, t > 0, x \in \mathbb{R} \tag{25}$$

The problem (24) can still be written as follows:

$$\begin{cases} {}^C D_t^\alpha u(x,t) = -\frac{\partial^4 u(x,t)}{\partial x^4} - 2\frac{\partial^2 u(x,t)}{\partial x^2} + N(u(x,t)) \\ u(x,0) = \cos x + \sin x \end{cases} \tag{26}$$

with

$$N(u(x,t)) = u^4(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} - u^2(x,t)\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^3 \tag{27}$$

the non-linear term

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Applying the reduced differential transformation to (25) gives :

$$\begin{cases} RDT\left({}^{C}D_t^{\alpha}u(x,t)\right) = -RDT\left(\frac{\partial^4 u(x,t)}{\partial x^4}\right) - 2RDT\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right) + RDT\left(N(u(x,t))\right) \\ RDT\left(u(x,0)\right) = \cos x + \sin x \end{cases} \tag{28}$$

This gives

$$\begin{cases} \frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x) = -\frac{\partial^4 U_k(x)}{\partial x^4} - 2\left(\frac{\partial^2 U_k(x)}{\partial x^2}\right) + N(U_k(x)), \forall k \geq 0 \\ U_0(x) = \cos x + \sin x \end{cases} \tag{29}$$

Using the idea of the principle of successive approximations, we obtain the following:

$$\begin{cases} \frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}^p(x) = -\frac{\partial^4 U_k^p(x)}{\partial x^4} - 2\left(\frac{\partial^2 U_k^p(x)}{\partial x^2}\right) + N(U_k^{p-1}(x)), \forall k \geq 0 \\ U_0^p(x) = \cos x + \sin x \end{cases} \quad \forall p \geq 1 \tag{30}$$

First iteration
For $p = 1$ we have :

$$\begin{cases} \frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}^1(x) = -\frac{\partial^4 U_k^1(x)}{\partial x^4} - 2\left(\frac{\partial^2 U_k^1(x)}{\partial x^2}\right) + N(U_k^0(x)), \forall k \geq 0 \\ U_0^1(x) = \cos x + \sin x \end{cases} \tag{31}$$

By Picard's principle ,there exists $U_k^0(x)$ any root of the equation $N(U_k^0(x)) = 0$
(30) becomes

$$\begin{cases} U_0^1(x) = \cos x + \sin x \\ U_{k+1}^1(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left(-\frac{\partial^4 U_k^1(x)}{\partial x^4} - 2\left(\frac{\partial^2 U_k^1(x)}{\partial x^2}\right)\right), \forall k \geq 0 \end{cases} \tag{32}$$

For $k = 0$, we have :

$$\begin{aligned} U_1^1(x) &= \frac{\Gamma(1)}{\Gamma(\alpha+1)}\left(-\frac{\partial^4 U_0^1(x)}{\partial x^4} - 2\left(\frac{\partial^2 U_0^1(x)}{\partial x^2}\right)\right) \\ &= \frac{\Gamma(1)}{\Gamma(\alpha+1)}\left(-\frac{\partial^4(\cos x+\sin x))}{\partial x^4} - 2\left(\frac{\partial^2(\cos x+\sin x)}{\partial x^2}\right)\right) \\ &= \frac{\cos x+\sin x}{\Gamma(\alpha+1)} \end{aligned} \tag{33}$$

For $k = 1$ we have:

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$$\begin{aligned} U_2^1(x) &= \tfrac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left(-\tfrac{\partial^4 U_1^1(x)}{\partial x^4}-2\left(\tfrac{\partial^2 U_1^1(x)}{\partial x^2}\right)\right)\\ &= \tfrac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left(\tfrac{1}{\Gamma(\alpha+1)}\right)\left(-\tfrac{\partial^4(\cos x+\sin x)}{\partial x^4}-2\left(\tfrac{\partial^2(\cos x+\sin x)}{\partial x^2}\right)\right)\\ &= \tfrac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left(\tfrac{1}{\Gamma(\alpha+1)}\right)(\cos x+\sin x)\\ &= \tfrac{\cos x+\sin x}{\Gamma(2\alpha+1)} \end{aligned} \tag{34}$$

For $k = 2$ we have:

$$\begin{aligned} U_3^1(x) &= \tfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left(-\tfrac{\partial^4 U_2^1(x)}{\partial x^4}-2\left(\tfrac{\partial^2 U_2^1(x)}{\partial x^2}\right)\right)\\ &= \tfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left(\tfrac{1}{\Gamma(2\alpha+1)}\right)\left(-\tfrac{\partial^4(\cos x+\sin x)}{\partial x^4}-2\left(\tfrac{\partial^2(\cos x+\sin x)}{\partial x^2}\right)\right)\\ &= \tfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left(\tfrac{1}{\Gamma(2\alpha+1)}\right)(\cos x+\sin x)\\ &= \tfrac{\cos x+\sin x}{\Gamma(3\alpha+1)} \end{aligned} \tag{35}$$

The inverse transform gives the approximate solution to the first ie

$$\begin{aligned} u^1(x,t) &= \sum_{k=0}^{\infty} U_k^1(x)t^{k\alpha}\\ &= U_0^1(x)t^0 + U_1^1(x)t^{\alpha} + U_2^1(x)t^{2\alpha} + U_3^1(x)t^{3\alpha} + \cdots\\ &= (\cos x+\sin x) + \frac{\cos x+\sin x}{\Gamma(\alpha+1)}t^{\alpha} + \frac{\cos x+\sin x}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{\cos x+\sin x}{\Gamma(3\alpha+1)}t^{3\alpha} + \cdots\\ &= (\cos x+\sin x)\left[1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right]\\ &= (\cos x+\sin x)\, E_{\alpha}(t^{\alpha}) \end{aligned} \tag{36}$$

Let's calculate $N(U_k^1(x))$
Like

$$\begin{aligned} N(u^1(x,t)) &= \left(u^1(x,t)\right)^4 \tfrac{\partial^2 u^1(x,t)}{\partial x^2} - \left(u^1(x,t)\right)^2\left(\tfrac{\partial^2 u^1(x,t)}{\partial x^2}\right)^3\\ &= \left((\cos x+\sin x)\,E_{\alpha}(t^{\alpha})\right)^4 \tfrac{\partial^2((\cos x+\sin x)E_{\alpha}(t^{\alpha}))}{\partial x^2} - \left((\cos x+\sin x)\,E_{\alpha}(t^{\alpha})\right)^2\left(\tfrac{\partial^2(\cos x+\sin x)E_{\alpha}(t^{\alpha}))}{\partial x^2}\right)^3\\ &= E_{\alpha}^5(t^{\alpha})\,(\cos x+\sin x)^4\tfrac{\partial^2(\cos x+\sin x)}{\partial x^2} - E_{\alpha}^5(t^{\alpha})\left((\cos x+\sin x)\right)^2\left(\tfrac{\partial^2(\cos x+\sin x)}{\partial x^2}\right)^3\\ &= -E_{\alpha}^5(t^{\alpha})\,(\cos x+\sin x)^5 + E_{\alpha}^5(t^{\alpha})\left((\cos x+\sin x)\right)^5\\ &= 0 \end{aligned} \tag{37}$$

So $N(U_k^1(x)) = 0$
Step 2 gives us the following algorithm

$$\begin{cases} U_{k+1}^2(x) = \tfrac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left(-\tfrac{\partial^4 U_k^2(x)}{\partial x^4}-2\left(\tfrac{\partial^2 U_k^2(x)}{\partial x^2}\right)+N(U_k^1(x))\right), \forall k \geq 0\\ \\ U_0^2(x) = \cos x+\sin x \end{cases} \tag{38}$$

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Since $N(U_k^1(x)) = 0$, we obtain

$$\begin{cases} U_0^2(x) = \cos x + \sin x \\ U_1^2(x) = \dfrac{\cos x + \sin x}{\Gamma(\alpha+1)} \\ U_2^2(x) = \dfrac{\cos x + \sin x}{\Gamma(2\alpha+1)} \\ U_3^2(x) = \dfrac{\cos x + \sin x}{\Gamma(3\alpha+1)} \\ \vdots \end{cases} \tag{39}$$

The solution to step 2 is

$$\begin{aligned} u^2(x,t) &= \sum_{k=0}^{\infty} U_k^2(x) t^{k\alpha} \\ &= U_0^2(x)t^0 + U_1^2(x)t^{\alpha} + U_2^2(x)t^{2\alpha} + U_3^2(x)t^{3\alpha} + \cdots \\ &= (\cos x + \sin x) + \frac{\cos x + \sin x}{\Gamma(\alpha+1)} t^{\alpha} + \frac{\cos x + \sin x}{\Gamma(2\alpha+1)} t^{2\alpha} + \frac{\cos x + \sin x}{\Gamma(3\alpha+1)} t^{3\alpha} + \cdots \\ &= (\cos x + \sin x) \left[1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \right] \\ &= (\cos x + \sin x) E_{\alpha}(t^{\alpha}) \end{aligned} \tag{40}$$

Subsequently, for $p \geq 3$, we can find the same approximate solution by recurrence at each step

$$u^3(x,t) = u^4(x,t) = u^5(x,t) = \cdots = u^p(x,t) = \cdots = (\cos x + \sin x) E_{\alpha}(t^{\alpha}) \tag{41}$$

The exact solution is therefore :

$$u(x,t) = \lim_{p \longrightarrow +\infty} u^p(x,t) = (\cos x + \sin x) E_{\alpha}(t^{\alpha}) \tag{42}$$

where $E_{\alpha}(t^{\alpha})$ is the Mittag-Leffler function.
For $\alpha = 1$,

$$u(x,t) = (\cos x + \sin x) E_1(t) = (\cos x + \sin x) e^t \tag{43}$$

is the exact solution to the problem

3.2 Example 2

Consider the following nonlinear fractional-order diffusion problem [15, 16]

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$$\begin{cases} {}^{C}D_t^{\alpha}u(x,t) = \frac{1}{2}\frac{\partial^2 u(x,t)}{\partial x^2} + u\left(x,t\right) + N\left(u(x,t)\right) \\ \\ u(x,0) = 2\cos x + 3\sin x \end{cases} \tag{44}$$

$$1 \quad < \quad \alpha \leq 1, t > 0, x \in \mathbb{R} \tag{45}$$

With

$$N\left(u(x,t)\right) = \left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right)^3 + \left(u(x,t)\frac{\partial^3 u(x,t)}{\partial x^3}\right)^3 \tag{46}$$

the non-linear term.

Using the fractional RDT method for the problem (43) we obtain :

$$\begin{cases} \frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x) = \frac{1}{2}\frac{\partial^2 U_k(x)}{\partial x^2} + U_k\left(x\right) + N\left(U_k(x)\right), \forall k \geq 0 \\ \\ U_0(x) = 2\cos x + 3\sin x \end{cases} \tag{47}$$

The method of successive approximations to (45) gives

$$\begin{cases} \frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}^p(x) = \frac{1}{2}\frac{\partial^2 U_k^p(x)}{\partial x^2} + U_k^p\left(x\right) + N\left(U_k^{p-1}(x)\right), \forall k \geq 0 \\ \\ U_0^p(x) = 2\cos x + 3\sin x \end{cases}, \forall p \geq 1 \tag{48}$$

Or

$$\begin{cases} U_{k+1}^p(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left(\frac{1}{2}\frac{\partial^2 U_k^p(x)}{\partial x^2} + U_k^p\left(x\right)\right) + \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}N\left(U_k^{p-1}(x)\right), \forall k \geq 0 \\ \\ U_0^p(x) = 2\cos x + 3\sin x \end{cases}, \forall p \geq 1 \tag{49}$$

Apply Picard's principle to (47) let $U_k^0(x)$ be a root of $N\left(U_k^0(x)\right) = 0$

Step 1

For $p = 1$, we obtain

$$\begin{cases} U_{k+1}^1(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left(\frac{1}{2}\frac{\partial^2 U_k^1(x)}{\partial x^2} + U_k^1\left(x\right)\right), \forall k \geq 0 \\ \\ U_0^1(x) = 2\cos x + 3\sin x \end{cases} \tag{50}$$

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For $k=0$, we have :

$$\begin{aligned} U_1^1(x) &= \frac{\Gamma(+1)}{\Gamma(\alpha+1)}\left(\frac{1}{2}\frac{\partial^2 U_0^1(x)}{\partial x^2} + U_0^1\left(x\right)\right) \\ &= \frac{\Gamma(+1)}{\Gamma(\alpha+1)}\left(\frac{1}{2}\frac{\partial^2(2\cos x+3\sin x)}{\partial x^2} + 2\cos x + 3\sin x\right) \\ &= \frac{1}{\Gamma(\alpha+1)}\left(\cos x + \frac{3}{2}\sin x\right) \\ &= \frac{\frac{1}{2}}{\Gamma(\alpha+1)}\left(2\cos x + 3\sin x\right) \end{aligned} \tag{51}$$

For $k=1$, we have :

$$\begin{aligned} U_2^1(x) &= \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left(\frac{1}{2}\frac{\partial^2 U_1^1(x)}{\partial x^2} + U_1^1\left(x\right)\right) \\ &= \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left(\frac{1}{\Gamma(\alpha+1)}\right)\left(\frac{1}{2}\frac{\partial^2\left(\left(\cos x+\frac{3}{2}\sin x\right)\right)}{\partial x^2} + \left(\cos x + \frac{3}{2}\sin x\right)\right) \\ &= \frac{\left(\frac{1}{2}\right)^2}{\Gamma(2\alpha+1)}\left(2\cos x + 3\sin x\right) \end{aligned} \tag{52}$$

For $k=2$, we have :

$$\begin{aligned} U_3^1(x) &= \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left(\frac{1}{2}\frac{\partial^2 U_2^1(x)}{\partial x^2} + U_2^1\left(x\right)\right) \\ &= \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left(\frac{1}{\Gamma(2\alpha+1)}\right)\left(\frac{1}{2}\frac{\partial^2\left(\frac{1}{2}\cos x+\frac{3}{4}\sin x\right)}{\partial x^2} + \left(\frac{1}{2}\cos x + \frac{3}{4}\sin x\right)\right) \\ &= \frac{\left(\frac{1}{2}\right)^3}{\Gamma(3\alpha+1)}\left(2\cos x + 3\sin x\right) \end{aligned} \tag{53}$$

By analogy for $k=n$ we obtain

$$U_n^1(x) = \frac{\left(\frac{1}{2}\right)^n}{\Gamma\left(n\alpha+1\right)}\left(2\cos x + 3\sin x\right), \forall n \geq 0 \tag{54}$$

The approximate solution to the first step is given by :

$$\begin{aligned} u^1(x,t) &= \sum_{n=0}^{\infty} U_n^1(x) t^{n\alpha} \\ &= \sum_{n=0}^{\infty}\left(\frac{\left(\frac{1}{2}\right)^n}{\Gamma(n\alpha+1)}\left(2\cos x + 3\sin x\right)\right) t^{n\alpha} \\ &= \left(2\cos x + 3\sin x\right)\sum_{n=0}^{\infty}\left(\frac{\left(\frac{1}{2}t^\alpha\right)^n}{\Gamma(n\alpha+1)}\right) \\ &= \left(2\cos x + 3\sin x\right) E_\alpha\left(\frac{1}{2}t^\alpha\right) \end{aligned} \tag{55}$$

Step 2

For $p=2$ (47) becomes :

$$\begin{cases} U_{k+1}^2(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left(\frac{1}{2}\frac{\partial^2 U_k^2(x)}{\partial x^2} + U_k^2\left(x\right)\right) + \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)} N\left(U_k^1(x)\right), \forall k \geq 0 \\ \\ U_0^2(x) = 2\cos x + 3\sin x \end{cases} \tag{56}$$

First, let's calculate $N\left(U_k^1(x)\right)$
or

$$\begin{aligned}
N\left(u^1(x,t)\right) &= \left(u^1(x,t)\frac{\partial u^1(x,t)}{\partial x}\right)^3 + \left(u^1(x,t)\frac{\partial^3 u^1(x,t)}{\partial x^3}\right)^3 \\
&= \left[\left((2\cos x + 3\sin x)\, E_\alpha\left(\tfrac{1}{2}t^\alpha\right)\right)\frac{\partial\left((2\cos x+3\sin x)E_\alpha\left(\frac{1}{2}t^\alpha\right)\right)}{\partial x}\right]^3 + \\
&\quad \left[\left((2\cos x + 3\sin x)\, E_\alpha\left(\tfrac{1}{2}t^\alpha\right)\right)\frac{\partial^3\left((2\cos x+3\sin x)E_\alpha\left(\frac{1}{2}t^\alpha\right)\right)}{\partial x^3}\right]^3 \\
&= \left[(2\cos x + 3\sin x)\frac{\partial(2\cos x+3\sin x)}{\partial x}\right]^3 E_\alpha^6\left(\tfrac{1}{2}t^\alpha\right) + \left[(2\cos x + 3\sin x)\frac{\partial^3(2\cos x+3\sin x)}{\partial x^3}\right]^3 E_\alpha^6\left(\tfrac{1}{2}t^\alpha\right) \\
&= \left[\left((2\cos x + 3\sin x)(3\cos x - 2\sin x)\right)^3 - \left((2\cos x + 3\sin x)(3\cos x - 2\sin x)\right)^3\right] E_\alpha^6\left(\tfrac{1}{2}t^\alpha\right) \\
&= 0
\end{aligned} \tag{57}$$

Since $N\left(u^1(x,t)\right) = 0$ then $N\left(U_k^1(x)\right) = 0$, we obtain the same algorithm as in step $p = 1$, i.e.

$$\begin{cases} U_{k+1}^2(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left(\frac{1}{2}\frac{\partial^2 U_k^2(x)}{\partial x^2} + U_k^2(x)\right), \forall k \geq 0 \\ \\ U_0^2(x) = 2\cos x + 3\sin x \end{cases} \tag{58}$$

The result is

$$U_n^2(x) = \frac{\left(\frac{1}{2}\right)^n}{\Gamma(n\alpha+1)}(2\cos x + 3\sin x), \forall n \geq 0 \tag{59}$$

The solution to the problem in step 2 is

$$\begin{aligned}
u^2(x,t) &= \sum_{n=0}^{\infty} U_n^2(x) t^{n\alpha} \\
&= \sum_{n=0}^{\infty}\left(\frac{\left(\frac{1}{2}\right)^n}{\Gamma(n\alpha+1)}(2\cos x + 3\sin x)\right)t^{n\alpha} \\
&= (2\cos x + 3\sin x)\sum_{n=0}^{\infty}\left(\frac{\left(\frac{1}{2}t^\alpha\right)^n}{\Gamma(n\alpha+1)}\right) \\
&= (2\cos x + 3\sin x)\, E_\alpha\left(\tfrac{1}{2}t^\alpha\right)
\end{aligned} \tag{60}$$

Subsequently, for $p \geq 3$, we can find the same approximate solution by recurrence at each step

$$u^3(x,t) = u^4(x,t) = u^5(x,t) = \cdots = u^p(x,t) = \cdots = (2\cos x + 3\sin x)\, E_\alpha\left(\frac{1}{2}t^\alpha\right) \tag{61}$$

The exact solution is therefore :

$$u(x,t) = \lim_{p\longrightarrow+\infty} u^p(x,t) = (2\cos x + 3\sin x)\, E_\alpha\left(\frac{1}{2}t^\alpha\right) \tag{62}$$

where $E_\alpha\left(\frac{1}{2}t^\alpha\right)$ is the Mittag-Leffler function.
The exact solution of (43) for $\alpha = 1$ is

$$u(x,t) = (2\cos x + 3\sin x)\, e^{\frac{1}{2}t} \tag{63}$$

3.3 Example 3

Consider the following Burgers-type problem [12, 13]:

$$\begin{cases} {}^{C}D_t^\alpha u(x,t) = \mu u(x,t) - u(x,t)\frac{\partial u(x,t)}{\partial x} - \frac{\omega}{2}\sin(2\omega x)\, e^{2\mu t} \\ \\ u(x,0) = \cos\omega x \end{cases}, \tag{64}$$

$$\omega \neq 0, 0 < \alpha \leq 1, t > 0, x \in \mathbb{R} \tag{65}$$

Let's ask

$$N(u(x,t)) = -u(x,t)\frac{\partial u(x,t)}{\partial x} - \frac{\omega}{2}\sin(2\omega x)\, e^{2\mu t} \tag{66}$$

the non-linear term

The problem (62) can still be written as

$$\begin{cases} {}^{C}D_t^\alpha u(x,t) = \mu u(x,t) + N(u(x,t)) \\ \\ u(x,0) = \cos\omega x, \omega \neq 0 \end{cases} \tag{67}$$

By applying the reduced differential transformation to (64) , we obtain the following iteration relation

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x) = \mu U_k(x) + N(U_k(x)) \tag{68}$$

with the initial condition in the form :

$$U_0(x) = \cos\omega x, \omega \neq 0 \tag{69}$$

In addition, the method of successive approximations gives

$$\begin{cases} U_0^p(x) = \cos\omega x, \omega \neq 0 \\ \\ U_{k+1}^p(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left(\mu U_k^p(x) + N(U_k^{p-1}(x))\right), \forall k \geq 0 \end{cases}, \forall p \geq 1 \tag{70}$$

Apply Picard's principle to (67) let $U_k^0(x)$ be any root of the equation $N(U_k^0(x)) = 0$

Thus for $p = 1$, we have

$$\begin{cases} U_0^1(x) = \cos\omega x, \omega \neq 0 \\ \\ U_{k+1}^1(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left(\mu U_k^1(x)\right), \forall k \geq 0 \end{cases} \tag{71}$$

The result is

$$\begin{cases} U_0^1(x) = \cos\omega x, \omega \neq 0 \\ \\ U_1^1(x) = \frac{\mu}{\Gamma(\alpha+1)}\cos\omega x \\ U_2^1(x) = \frac{\mu^2}{\Gamma(2\alpha+1)}\cos\omega x \\ \\ U_n^1(x) = \frac{\mu^n}{\Gamma(n\alpha+1)}\cos\omega x \end{cases} \tag{72}$$

The approximate solution at step $p = 1$ is

$$\begin{cases} u^1(x,t) = \sum_{n=0}^{\infty} U_n^1(x) t^{n\alpha} \\ \qquad = \sum_{n=0}^{\infty}\left(\frac{\mu^n}{\Gamma(n\alpha+1)}\cos\omega x\right) t^{n\alpha} \\ \qquad = \sum_{n=0}^{\infty}\left(\frac{(\mu t^\alpha)^n}{\Gamma(n\alpha+1)}\right)\cos\omega x \\ \qquad = E_\alpha\left(\mu t^\alpha\right)\cos\omega x \end{cases} \tag{73}$$

Let's calculate $N(U_k^1(x))$
Like

$$\begin{aligned} N(u^1(x,t)) &= -u^1(x,t)\frac{\partial u^1(x,t)}{\partial x} - \frac{\omega}{2}\sin(2\omega x)E_\alpha^2(\mu t^\alpha) \\ &= -E_\alpha(\mu t^\alpha)\cos\omega x\frac{\partial(E_\alpha(\mu t^\alpha)\cos\omega x)}{\partial x} - \frac{\omega}{2}\sin(2\omega x)E_\alpha^2(\mu t^\alpha) \\ &= -\left(E_\alpha(\mu t^\alpha)\right)^2\cos\omega x\frac{\partial(\cos\omega x)}{\partial x} - \frac{\omega}{2}\sin(2\omega x)\left(E_\alpha(\mu t^\alpha)\right)^2 \\ &= -\left(E_\alpha(\mu t^\alpha)\right)^2\omega\cos\omega x\sin x\omega - \frac{\omega}{2}\sin(2\omega x)\left(E_\alpha(\mu t^\alpha)\right)^2 \\ &= \left(E_\alpha(\mu t^\alpha)\right)^2\left(\frac{\omega}{2}\sin(2\omega x) - \frac{\omega}{2}\sin(2\omega x)\right) \\ &= 0 \end{aligned} \tag{74}$$

Since $N\left(u^1(x,t)\right) = 0$ then $N\left(U_k^1(x)\right) = 0$, we obtain the same algorithm as in step $p = 1$, i.e.

$$\begin{cases} U_0^2(x) = \cos\omega x, \omega \neq 0 \\ \\ U_{k+1}^2(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left(\mu U_k^2(x)\right), \forall k \geq 0 \end{cases} \tag{75}$$

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then after calculations we have :

$$\begin{cases} U_0^2(x) = \cos\omega x, \omega \neq 0 \\ \\ U_1^2(x) = \frac{\mu}{\Gamma(\alpha+1)}\cos\omega x \\ U_2^2(x) = \frac{\mu^2}{\Gamma(2\alpha+1)}\cos\omega x \\ \\ U_n^2(x) = \frac{\mu^n}{\Gamma(n\alpha+1)}\cos\omega x \end{cases} \tag{76}$$

The solution to the problem in step 2 is

$$\begin{aligned} u^2(x,t) &= \sum_{n=0}^{\infty} U_n^1(x) t^{n\alpha} \\ &= \sum_{n=0}^{\infty} \left(\frac{\mu^n}{\Gamma(n\alpha+1)}\cos\omega x\right) t^{n\alpha} \\ &= \sum_{n=0}^{\infty} \left(\frac{(\mu t^\alpha)^n}{\Gamma(n\alpha+1)}\right)\cos\omega x \\ &= E_\alpha(\mu t^\alpha)\cos\omega x \end{aligned} \tag{77}$$

Subsequently, for $p \geq 3$, we can find the same approximate solution by recurrence at each step

$$u^3(x,t) = u^4(x,t) = u^5(x,t) = \cdots = u^p(x,t) = \cdots = E_\alpha(\mu t^\alpha)\cos\omega x \tag{78}$$

The exact solution is therefore :

$$u(x,t) = \lim_{p \longrightarrow +\infty} u^p(x,t) = E_\alpha(\mu t^\alpha)\cos\omega x \tag{79}$$

where $E_\alpha(\mu t^\alpha)$ is the Mittag-Leffler function.

The exact solution of (43) for $\alpha = 1$ is

$$u(x,t) = E_1(\mu t)\cos\omega x = e^{\mu t}\cos\omega x \tag{80}$$

3.4 Example 4

Consider the following nonlinear fractional-order diffusion problem [12, 14]

$$\begin{cases} {}^C D_t^\alpha u(x,t) = \lambda \frac{\partial^2 u(x,t)}{\partial x^2} + \omega^6 u^3(x,t) + \left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^3 \\ \\ u(x,0) = \cos(\omega x) + \sin(\omega x) \end{cases}, \tag{81}$$

$$\omega \neq 0, 1 < \alpha \leq 1, t > 0, x \in \mathbb{R} \tag{82}$$

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Let's ask

$$N(u(x,t)) = \omega^6 u^3(x,t) + \left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^3 \tag{83}$$

the non-linear term

$$\begin{cases} {}^C D_t^{\alpha} u(x,t) = \lambda \frac{\partial^2 u(x,t)}{\partial x^2} + N(u(x,t)) \\ \\ u(x,0) = \cos(\omega x) + \sin(\omega x) \end{cases}, \omega \neq 0 \tag{84}$$

Applying the reduced differential transformation to (80) , we obtain the following iteration relation

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)} U_{k+1}(x) = \lambda \frac{\partial^2 U_k(x)}{\partial x^2} + N(U_k(x)) \tag{85}$$

with the initial condition in the form :

$$U_0(x) = \cos(\omega x) + \sin(\omega x), \omega \neq 0 \tag{86}$$

In addition, the method of successive approximations gives

$$\begin{cases} U_{k+1}^p(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)} \left(\lambda \frac{\partial^2 U_k^p(x)}{\partial x^2} + N(U_k^{p-1}(x))\right) \\ \\ U_0^p(x) = \cos(\omega x) + \sin(\omega x), \omega \neq 0 \end{cases}, \forall p \geq 1 \tag{87}$$

Step 1

For $p = 1$ we have :

$$\begin{cases} U_{k+1}^1(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)} \left(\lambda \frac{\partial^2 U_k^1(x)}{\partial x^2} + N(U_k^0(x))\right) \\ \\ U_0^1(x) = \cos(\omega x) + \sin(\omega x), \omega \neq 0 \end{cases} \tag{88}$$

Apply Picard's principle to (84), let $U_k^0(x)$ be any root of the equation $N(U_k^0(x)) = 0$ then (84) becomes

$$\begin{cases} U_0^1(x) = \cos(\omega x) + \sin(\omega x), \omega \neq 0 \\ \\ U_{k+1}^1(x) = \lambda \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)} \frac{\partial^2 U_k^1(x)}{\partial x^2} \end{cases} \tag{89}$$

Using (85), we obtain the following values in succession

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$$\begin{cases} U_0^1(x) = \cos(\omega x) + \sin(\omega x), \omega \neq 0 \\ U_1^1(x) = \frac{\left(-\omega^2\lambda\right)^1}{\Gamma(\alpha+1)}(\cos x\omega + \sin x\omega) \\ U_2^1(x) = \frac{\left(-\omega^2\lambda\right)^2}{\Gamma(2\alpha+1)}(\cos x\omega + \sin x\omega) \\ U_3^1(x) = \frac{\left(-\omega^2\lambda\right)^3}{\Gamma(3\alpha+1)}(\cos x\omega + \sin x\omega) \\ \vdots \\ U_n^1(x) = \frac{\left(-\omega^2\lambda\right)^n}{\Gamma(n\alpha+1)}(\cos x\omega + \sin x\omega) \end{cases} \tag{90}$$

The approximate solution at step $p = 1$ is

$$\begin{cases} u^1(x,t) = \sum_{n=0}^{\infty} U_n^1(x) t^{n\alpha} \\ \qquad = \sum_{n=0}^{\infty} \left(\frac{\left(-\omega^2\lambda\right)^n}{\Gamma(n\alpha+1)} (\cos x\omega + \sin x\omega) \right) t^{n\alpha} \\ \qquad = \sum_{n=0}^{\infty} \left(\frac{\left(-\lambda\omega^2 t^\alpha\right)^n}{\Gamma(n\alpha+1)} \right) (\cos x\omega + \sin x\omega) \\ \qquad = (\cos x\omega + \sin x\omega) E_\alpha\left(-\lambda\omega^2 t^\alpha\right) \end{cases} \tag{91}$$

Step 2

For $p = 2$ (83) becomes

$$\begin{cases} U_{k+1}^2(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)} \left(\lambda \frac{\partial^2 U_k^2(x)}{\partial x^2} + N(U_k^1(x)) \right) \\ U_0^2(x) = \cos(\omega x) + \sin(\omega x), \omega \neq 0 \end{cases} \tag{92}$$

Let's calculate $N(U_k^1(x))$

Since

$$\begin{aligned} N(u^1(x,t)) &= \omega^6 \left(u^1(x,t)\right)^3 + \left(\frac{\partial^2 u^1(x,t)}{\partial x^2}\right)^3 \\ &= \omega^6 \left((\cos x\omega + \sin x\omega) E_\alpha\left(-\lambda\omega^2 t^\alpha\right)\right)^3 + \left(\frac{\partial^2 (\cos x\omega + \sin x\omega) E_\alpha\left(-\lambda\omega^2 t^\alpha\right)}{\partial x^2}\right)^3 \\ &= \left(E_\alpha\left(-\lambda\omega^2 t^\alpha\right)\right)^3 \omega^6 \left((\cos x\omega + \sin x\omega)\right)^3 + \left(E_\alpha\left(-\lambda\omega^2 t^\alpha\right)\right)^3 \left(\frac{\partial^2 (\cos x\omega + \sin x\omega)}{\partial x^2}\right)^3 \\ &= \left(E_\alpha\left(-\lambda\omega^2 t^\alpha\right)\right)^3 \omega^6 \left((\cos x\omega + \sin x\omega)\right)^3 + \left(E_\alpha\left(-\lambda\omega^2 t^\alpha\right)\right)^3 \left(-\omega^2 (\cos x\omega + \sin x\omega)\right)^3 \\ &= \left(E_\alpha\left(-\lambda\omega^2 t^\alpha\right)\right)^3 \left[\omega^6 \left((\cos x\omega + \sin x\omega)\right)^3 - \left(\omega^2 (\cos x\omega + \sin x\omega)\right)^3\right] \\ &= 0 \end{aligned} \tag{93}$$

then $N(U_k^1(x)) = 0$

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This gives

$$\begin{cases} U_0^2(x) = \cos(\omega x) + \sin(\omega x), \omega \neq 0 \\ U_1^2(x) = \frac{\left(-\omega^2\lambda\right)^1}{\Gamma(\alpha+1)}(\cos x\omega + \sin x\omega) \\ U_2^2(x) = \frac{\left(-\omega^2\lambda\right)^2}{\Gamma(2\alpha+1)}(\cos x\omega + \sin x\omega) \\ U_3^2(x) = \frac{\left(-\omega^2\lambda\right)^3}{\Gamma(3\alpha+1)}(\cos x\omega + \sin x\omega) \\ \vdots \\ U_n^2(x) = \frac{\left(-\omega^2\lambda\right)^n}{\Gamma(n\alpha+1)}(\cos x\omega + \sin x\omega) \end{cases} \tag{94}$$

The solution to the problem at step $p = 2$ is

$$\begin{aligned} u^2(x,t) &= \sum_{n=0}^{\infty} U_n^2(x) t^{n\alpha} \\ &= \sum_{n=0}^{\infty} \left(\frac{\left(-\omega^2\lambda\right)^n}{\Gamma(n\alpha+1)} (\cos x\omega + \sin x\omega) \right) t^{n\alpha} \\ &= \sum_{n=0}^{\infty} \left(\frac{\left(-\lambda\omega^2 t^\alpha\right)^n}{\Gamma(n\alpha+1)} \right) (\cos x\omega + \sin x\omega) \\ &= (\cos x\omega + \sin x\omega) E_\alpha\left(-\lambda\omega^2 t^\alpha\right) \end{aligned} \tag{95}$$

so

$$u^1(x,t) = u^2(x,t) = (\cos x\omega + \sin x\omega) E_\alpha\left(-\lambda\omega^2 t^\alpha\right) \tag{96}$$

Subsequently, for $p \geq 3$, we can find the same approximate solution by recurrence at each step

$$u^3(x,t) = u^4(x,t) = u^5(x,t) = \cdots = u^p(x,t) = \cdots = (\cos x\omega + \sin x\omega) E_\alpha\left(-\lambda\omega^2 t^\alpha\right) \tag{97}$$

The exact solution is therefore :

$$u(x,t) = \lim_{p \longrightarrow +\infty} u^p(x,t) = (\cos x\omega + \sin x\omega) E_\alpha\left(-\lambda\omega^2 t^\alpha\right) \tag{98}$$

where $E_\alpha\left(-\lambda\omega^2 t^\alpha\right)$ is the Mittag-Leffler function.
The exact solution of (78) for $\alpha = 1$ is

$$u(x,t) = (\cos x\omega + \sin x\omega) e^{-\lambda\omega^2 t} \tag{99}$$

4 Conclusion

In this article, we first describe the BYJ numerical method and use it to solve certain nonlinear fractional-time partial differential equations in the sense of Caputo. The BYJ method was used to solve four problems whose difficulty levels depended on the nonlinear term introduced, and led to more accurate results. The results obtained show that the new method is efficient and easy to use for finding approximate solutions for fractional-order partial differential equations in time. Thus, the proposed method has a significant impact on the way engineering, physics and other disciplines solve fractional-order equations.

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