Investigations of Wet Dark Fluid in Modified Theory of Gravitation for Bianchi Type V

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Abstract

In this paper we present the model of five dimensional Bianchi type V space-time with Wet Dark Fluid in presence of $f(R,T)$ theory of gravitation. We calculate the solution of the field equation from two volumetric expansion model as power law expansion model and exponential expansion model. Also we studied the physical parameters of the model. To find out the expansion history of the universe and for determining whether the expansion is accelerating or decelerating, along with the associated rate we discussed energy conditions, stability of the model, statefinder parameter, look-back time, proper distance, luminosity distance, angular diameter, cosmic snap parameter and jerk parameter with redshift .

Keywords: Bianchi type V, Deceleration parameter, $f(R,T)$ theory of gravitation, Wet dark fluid.

1. Introduction

The motion and behavior of large-scale cosmic structure such as galaxies, clusters, stars, and planets are influenced by gravity. This concept of gravity is essential for understanding the stretching and twisting of space-time. The formulation of gravity in terms of curvature has fundamentally shifted our understanding of the cosmos, which is now known to consist of two essential and mysterious components: dark energy (DE) and dark matter (DM). Both DE and DM play critical roles in the expansion of the universe. Dark energy, which acts as a repulsive force countering gravity, is responsible for the universe's accelerated expansion. However, its true nature remains elusive. Dark energy, with its negative pressure, is recognized as the force behind the accelerated expansion of the universe. This behavior has prompted analytical research into whether dark energy is the sole cause of the universe's acceleration, or if other cosmic factors may contribute to this phenomenon. Such uncertainties have spurred efforts to explore alternative perspectives on general relativity. Since DE does not interact directly with baryonic matter, it cannot be detected through conventional means. Within the framework of general relativity (GR), DE is modeled as an isotropic fluid with constant energy density and negative pressure. Among various candidates, the cosmological constant (Λ) remains the leading explanation due to its consistency with a constant energy density throughout cosmic evolution. Extensive research has been conducted to deepen our understanding of the dark-energy-driven late-time acceleration of the universe's expansion. Observational evidence gathered from high-redshift Type Ia supernovae, notably by the High-z Supernova Search Team, has revealed that our universe is undergoing an accelerated expansion. The observations indicate that the model best fitting the data features a nonzero cosmological constant and a negative deceleration parameter, as inferred from luminosity measurements of numerous galaxy clusters. This claim has gained substantial support through further evidence provided by advancements in Type Ia supernova studies, cosmic microwave background radiation (CMBR) anisotropies, large-scale structure surveys, baryon acoustic oscillations (BAO), and weak gravitational lensing observations. These diverse datasets collectively suggest that the present universe is geometrically flat. Perhaps the most striking conclusion from these findings is the current energy composition of the universe is approximately consists of 4.6% baryonic (non-relativistic) matter, about 24% is attributed to non-baryonic dark matter (DM), and the remaining 71.4% is made up of an exotic form of energy with negative pressure, known as dark energy (DE).

The researchers have proposed several dynamical DE models characterized by an evolving effective equation-of-state (EoS) parameter, w. Despite these efforts, the exact value of w remains uncertain, allowing for multiple viable DE candidates. Besides the ΛCDM model (with $w = -1$), alternative scalar field models have been explored, including quintessence $(-\frac{2}{3} \leq w \leq -\frac{1}{3})$, phantom fields (w), k-essence, tachyons, and quintom models. Interacting DE models have also been developed, such as the Chaplygin gas model, holographic dark energy etc. There is new candidate for dark energy introduced by (Holman & Naidu, 2004) called Wet Dark Fluid (WDF). Wet Dark Fluid model is in the spirit of the generalized Chaplygin gas (Gorini, et al., 2005). The Wet dark fluid cosmological model for dark energy which derive from an experimental equation of state proposed by (Hayward, 1967) and (Tait, 1988) to treat aqueous solutions and water. For Wet dark fluid the equation of state is written as

$$p_{WDF} = \gamma(\rho_{WDF} - \rho^*), \tag{1.1}$$

where p_{WDF} is pressure of WDF and ρ_{WDF} represents energy density of WDF. In this equation the parameters $\rho^*$and γ are taken to be positive and we restrict our analysis to the range $0 < \gamma < 1.$ This form is motivated by its applicability to real fluid such as water, where intermolecular attraction at small scales can results in negative pressure. Here $\gamma = {c_s}^2$ given by (Babichev, et al., 2004) and c_s denotes the adiabatic sound speed in wet dark fluid. In literature many researcher (Adhav, et al., 2011), (Ravishankar, et al., 2013), (Sarkar, 2015), (Chirde & Shekh , 2016), (Mahanta, 2017), (Samanta & Bishi, 2017), (Angit , et al., 2019), (Singh, et al., 2021), (Shobhane & Deo, 2021), (Pawar, et al., 2024)investigated wet dark fluid cosmological model.

General relativity, in its classical form, predicts the existence of singularities, a concept that poses challenges to cosmology, particularly in relation to the Big Bang theory. The absence of singularities in models allows researchers to predict the amount of pressure that would accumulate during both the decelerating and accelerating phases of cosmic expansion. General relativity itself remains a remarkably robust and mathematically it is theory of gravity. However, it is inadequate when it comes to explaining the nature of dark energy and dark matter, prompting cosmologists to look beyond GR for potential modifications. These modifications to GR have emerged in two primary ways. The first involves altering Einstein's field equations. The second approach introduces modified theories of gravity, such as scalar-tensor theories, which seek to adjust the left-hand side of Einstein's equation. Each of these modifications proposes a new theory of gravitation, with several variations emerging in recent years. In response to these challenges, alternative approaches have been developed that involve modifying the geometric sector of Einstein's field equations. These modifications lead to extended theories of gravity, such as $f(R)$ gravity, $f(T)$ gravity, $f(G)$ gravity, and $f(R,T)$ gravity. These frameworks provide a means to explain the accelerated expansion of the universe without invoking exotic dark energy components. By modifying the standard gravitational theory, these models offer promising avenues for resolving the outstanding issues associated with dark energy and may ultimately provide deeper insights into the structure and evolution of the universe. Among these, the $f(R,T)$ gravity theory is the most well-known and widely studied. In this study, we investigate the dynamics of the universe within the framework of $f(R,T)$ gravity, a modified theory of gravity proposed as an extension of General Relativity, where the gravitational Lagrangian is taken to be a general function of the Ricci scalar and the trace of the energy-momentum tensor. This approach introduces a coupling between matter and geometry, which allows for new gravitational effects that may account for the observed late-time cosmic acceleration. This theory has become particularly important for addressing the dark energy problem and has garnered significant attention over the past decade. Despite this, the unification of early-time inflation and late-time acceleration has been successfully explored within the context of viable $f(R,T)$ gravity models. Several researcher (Shamir, 2015), (Mete & Mule, 2017), (Shaikh & Wankhade, 2017), (Chundawat & Mehta, 2021), (Pradhan, et al., 2023), (Thakare, et al., 2023) studied $f(R,T)$ theory of gravity.

In cosmology investigations of higher dimensional space-time plays a vital role to describe the early stages of evolution of the universe. The study of higher-dimensional cosmological models has gained importance as a compelling approach to understanding the fundamental nature of the universe. Motivated by developments in unified field theories, such as string theory, M-theory, and supergravity, the concept that our observable four-dimensional spacetime may be a submanifold embedded in a higher-dimensional manifold has become an essential aspect of modern theoretical physics. Higher dimensions provide a framework where all interactions arise from geometry of the extra dimension space. It is also believed that the interaction of particles may be well explained by higher dimensional space-time. Higher dimensions offers deeper explanations and unification of fundamental phenomena and also alternative explanations for cosmic acceleration or dark matter effects. Inflation, initial singularity and anisotropies are better described with extra dimensions. Anisotropic higher dimensional models allow studying the isotropization process of the early universe and the dimensional compactification. In the literature (Xu & Lu, 2005), (Aguilar & Romero, 2009) , (Reddy & Naidu, 2009), (Samanta, et al., 2014) , (Sahoo, 2017), (Pawar, et al., 2023) , (Thakre, et al., 2024), (Daimary & Baruah, 2024), (Oli & Santhosh, 2025) are some of the authors who have studied higher dimensional cosmological model in various theories. In $f(R,T)$ theory of gravity, the inclusion of higher-dimensional corrections leads to non-trivial modifications of the field equations, potentially accounting for the late-time acceleration of the universe without invoking exotic matter fields. Motivated from above work, in this paper, we studied anisotropic five dimensional Bianchi type V space-time filled with wet dark fluid within the framework of $f(R,T)$ theory of gravitation. This paper contains four sections, in section 2, contains brief information about $f(R,T)$ theory of gravity. In section 3, metric and field equations, section 4 contains solutions of five dimensional Bianchi type V space-time and physical parameter of the model and in section 5 includes the conclusion of the study.

2. Basic formalism of $f(R,T)$ Theory

The field equation for $f(R,T)$ theory of gravity is derived from the action

$$S=\frac{1}{16\pi}\int[f(R,T)+L_m]\sqrt{-g}d^4x \tag{2.1}$$

Where $f(R,T)$ is arbitrary function of Ricci scalar R and trace T of energy momentum tensor T_{ij} and L_m is the matter lagrangian.

The energy momentum tensor T_{ij} is defined as

$$T_{ij}=-\frac{2\delta(\sqrt{-g}\,L_m)}{\sqrt{-g}\,\delta g_{ij}} \tag{2.2}$$

Here we consider that the dependence of matter Lagrangian is merely on g_{ij} i.e. metric tensor rather than on its derivatives.

$$T_{ij}=L_m g_{ij}-2\frac{\delta L_m}{\delta g^{ij}} \tag{2.3}$$

Now by varying the action S in equation (2.1) with respect to the metric tensor g_{ij} the field equations of $f(R,T)$ gravity are obtained as

$$f_R(R,T)R_{ij}-\frac{1}{2}f(R,T)g_{ij}-\left(\nabla_i\nabla_j-g_{ij}\Box\right)f_R(R,T)=8\pi T_{ij}-f_T(R,T)(T_{ij}+\theta_{ij}) \tag{2.4}$$

Where ∇_i denotes the covariant derivative and $\Box=\nabla^i\nabla_j$, $\theta_{ij}=g^{lk}\frac{\delta T_{lk}}{\delta g^{ij}}$.

Now contracting eq. (2.4) we obtained

$$f_R(R,T)R+3\,\Box f_R(R,T)-2f(R,T)=8\pi T-f_T(R,T)(T+\theta) \tag{2.5}$$

Where $\theta = \theta_i^{\ i}$. This equation is important because it gives a relation between Ricci scalar and trace T of the energy momentum tensor.

In this paper we assume the energy momentum tensor for wet dark fluid T_{ij} is

$$T_{ij} = (p_{WDF} + \rho_{WDF})\, u_i\, u_j + p_{WDF} g_{ij} \tag{2.6}$$

With $g_{ij}u^i u^j = 1$ (2.7)

where p_{WDF} is pressure of the wet dark fluid, ρ_{WDF} is the density of wet dark fluid and u_i is five velocity vector of the fluid. Here we assume the matter Lagrangian as $L_m = -p$ which gives

$$\theta_{ij} = -pg_{ij} - 2T_{ij} \tag{2.8}$$

And then the field equation takes the form

$$f_R(R,T)R_{ij} - \frac{1}{2}f(R,T)g_{ij} - (\nabla_i\nabla_j - g_{ij}\,\Box)f_R(R,T) = 8\pi T_{ij} f_T(R,T)(T_{ij} + pg_{ij}) \tag{2.9}$$

It may be mentioned here that these field equations depend on the physical nature of the matter field. Many theoretical models are possible corresponding to different matter contribution for $f(R,T)$ gravity. However, (Harko, et al., 2011) gave three different classes of these model:

$$f(R,T) = \begin{cases} R + 2f(T) \\ f_1(R) + f_2(T) \\ f_1(R) + f_2(R)f_3(T) \end{cases} \tag{2.10}$$

In this article we are studied the case $f(R,T) = R + 2f(T)$. In this particular case, the form of field equation is

$$R_{ij} - \frac{1}{2}Rg_{ij} = 8\pi T_{ij} + 2f'(T)T_{ij} + [f(T) + 2pf'(T)]g_{ij} \tag{2.11}$$

Here we choose $f(T) = \mu T$.

3. Metric and field equations

The five dimensional Bianchi type V space-time is

$$ds^2 = dt^2 - P_1^2 dx^2 - P_2^2\, e^{-2mx} dy^2 - P_3^2\, e^{-2mx} dz^2 - P_4^2 e^{-2mx} du^2 \tag{3.1}$$

Where P_1, P_2, P_3, P_4 are the functions of cosmic time t only and m is constant.

From (3.1), (2.6) and (2.7), equation (2.11) becomes

$$\frac{\ddot{P}_2}{P_2} + \frac{\ddot{P}_3}{P_3} + \frac{\ddot{P}_4}{P_4} + \frac{\dot{P}_2}{P_2}\frac{\dot{P}_3}{P_3} + \frac{\dot{P}_2}{P_2}\frac{\dot{P}_4}{P_4} + \frac{\dot{P}_3}{P_3}\frac{\dot{P}_4}{P_4} - \frac{3m^2}{P_1^{\ 2}} = 8\pi p_{WDF} + 10\mu p_{WDF} + \mu\rho_{WDF} \tag{3.2}$$

$$\frac{\ddot{P}_1}{P_1} + \frac{\ddot{P}_3}{P_3} + \frac{\ddot{P}_4}{P_4} + \frac{\dot{P}_1}{P_1}\frac{\dot{P}_3}{P_3} + \frac{\dot{P}_1}{P_1}\frac{\dot{P}_4}{P_4} + \frac{\dot{P}_3}{P_3}\frac{\dot{P}_4}{P_4} - \frac{3m^2}{P_1^{\ 2}} = 8\pi p_{WDF} + 10\mu p_{WDF} + \mu\rho_{WDF} \tag{3.3}$$

$$\frac{\ddot{P}_1}{P_1} + \frac{\ddot{P}_2}{P_2} + \frac{\ddot{P}_4}{P_4} + \frac{\dot{P}_1}{P_1}\frac{\dot{P}_2}{P_2} + \frac{\dot{P}_1}{P_1}\frac{\dot{P}_4}{P_4} + \frac{\dot{P}_2}{P_2}\frac{\dot{P}_4}{P_4} - \frac{3m^2}{P_1^{\ 2}} = 8\pi p_{WDF} + 10\mu p_{WDF} + \mu\rho_{WDF} \tag{3.4}$$

$$\frac{\ddot{P}_1}{P_1} + \frac{\ddot{P}_2}{P_2} + \frac{\ddot{P}_3}{P_3} + \frac{\dot{P}_1}{P_1}\frac{\dot{P}_2}{P_2} + \frac{\dot{P}_1}{P_1}\frac{\dot{P}_3}{P_3} + \frac{\dot{P}_2}{P_2}\frac{\dot{P}_4}{P_4} - \frac{3m^2}{P_1^{\ 2}} = 8\pi p_{WDF} + 10\mu p_{WDF} + \mu\rho_{WDF} \tag{3.5}$$

$$\frac{\dot{P}_1}{P_1}\frac{\dot{P}_2}{P_2}+\frac{\dot{P}_1}{P_1}\frac{\dot{P}_3}{P_3}+\frac{\dot{P}_1}{P_1}\frac{\dot{P}_4}{P_4}+\frac{\dot{P}_2}{P_2}\frac{\dot{P}_3}{P_3}+\frac{\dot{P}_2}{P_2}\frac{\dot{P}_4}{P_4}+\frac{\dot{P}_3}{P_3}\frac{\dot{P}_4}{P_4}-\frac{6m^2}{{P_1}^2}=(16\pi+12\mu)p_{WDF}+(8\pi+3\mu)\rho_{WDF} \quad (3.6)$$

$$3\frac{\dot{P}_1}{P_1}-\frac{\dot{P}_2}{P_2}-\frac{\dot{P}_3}{P_3}-\frac{\dot{P}_4}{P_4}=0 \quad (3.7)$$

Here the overhead dot on the P_1, P_2, P_3, P_4 denotes ordinary differentiation with respect to time t.

4. Solution of field equations and physical quantities of the model

Integrating equation (3.7) we get

$${P_1}^3 = P_2P_3P_4$$

$$P_1 = (P_2P_3P_4)^{\frac{1}{3}} \quad (4.1)$$

Let V be the function of time t defined by

$$V = P_1P_2P_3P_4 = {P_1}^4 \quad (4.2)$$

Using (4.1) and (4.2) in (3.2)-(3.6) we get

$$P_1(t) = D_1V^{\frac{1}{4}}\exp(X_1\int\frac{dt}{V}) \quad (4.3)$$

$$P_2(t) = D_2V^{\frac{1}{4}}\exp(X_2\int\frac{dt}{V}) \quad (4.4)$$

$$P_3(t) = D_3V^{\frac{1}{4}}\exp(X_3\int\frac{dt}{V}) \quad (4.5)$$

$$P_4(t) = D_4V^{\frac{1}{4}}\exp(X_4\int\frac{dt}{V}) \quad (4.6)$$

Where integrating constants D_i $(i = 1,2,3,4)$ and X_i $(i = 1,2,3,4)$ satisfying $D_1D_2D_3D_4 = 1$ and $X_1 + X_2 + X_3 + X_4 = 0$.

The field equations (3.2)-(3.6) are differential equations with unknowns $P_1, P_2, P_3, P_4, p_{WDF}, \rho_{WDF}$. To determine the complete solution of above equations we require one physical condition. Hence we use volumetric expansion Laws.

i) Power law $V = Mt^{4n}$, M is constant.

ii) Exponential law $V = Lt^{4N}$, L and N are constant.

The Hubble parameter H is

$$H = \frac{1}{4}(H_x + H_y + H_z + H_u) \quad (4.7)$$

Where H_x, H_y, H_z, H_u are the directional Hubble's parameters along X, Y, Z, U axis respectively.

$$H_x = \frac{\dot{P}_1}{P_1},\ H_y = \frac{\dot{P}_2}{P_2},\ H_z = \frac{\dot{P}_3}{P_3},\ H_u = \frac{\dot{P}_4}{P_4} \quad (4.8)$$

The Average anisotropic expansion parameter A_m, shear scalar σ^2 and expansion scalar θ is

$$A_m = \frac{1}{4}\sum_{i=1}^{4}(\frac{\Delta H_i}{H})^2, \qquad \Delta H_i = H_i - H \tag{4.9}$$

$$\sigma^2 = \frac{1}{2}\left(\sum_{i=1}^{4} H_i^2 - 3H^2\right) = \frac{1}{2}\left({H_x}^2 + {H_y}^2 + {H_z}^2 + {H_u}^2 - \frac{\theta^2}{4}\right) \tag{4.10}$$

$$\theta = \frac{\dot{P}_1}{P_1} + \frac{\dot{P}_2}{P_2} + \frac{\dot{P}_3}{P_3} + \frac{\dot{P}_4}{P_4} \tag{4.11}$$

Average scale factor is denoted by R and is given by

$$R = (V)^{\frac{1}{4}} \tag{4.12}$$

4.1 Power Law Model

$$V = Mt^{4n}, \text{ where } M \text{ is constant.} \tag{4.1.1}$$

Using (4.1.1) in (4.3) – (4.4) we get

$$P_1(t) = D_1 M^{\frac{1}{4}} t^n \exp[\frac{X_1 t^{1-4n}}{M(1-4n)}] \tag{4.1.2}$$

$$P_2(t) = D_2 M^{\frac{1}{4}} t^n \exp[\frac{X_2 t^{1-4n}}{M(1-4n)}] \tag{4.1.3}$$

$$P_3(t) = D_3 M^{\frac{1}{4}} t^n \exp[\frac{X_3 t^{1-4n}}{M(1-4n)}] \tag{4.1.4}$$

$$P_4(t) = D_4 M^{\frac{1}{4}} t^n \exp[\frac{X_4 t^{1-4n}}{M(1-4n)}] \tag{4.1.5}$$

where $D_1 D_2 D_3 D_4 = 1$ and $X_1 + X_2 + X_3 + X_4 = 0$.

Then the solution is

$$ds^2 = dt^2 - \left\{D_1 M^{\frac{1}{4}} t^n \exp\left[\frac{X_1 t^{1-4n}}{M(1-4n)}\right]\right\}^2 dx^2 - e^{-2mx}\left\{D_2 M^{\frac{1}{4}} t^n \exp[\frac{X_2 t^{1-4n}}{M(1-4n)}]\right\}^2 dy^2 - e^{-2mx}\left\{D_3 M^{\frac{1}{4}} t^n \exp[\frac{X_3 t^{1-4n}}{M(1-4n)}]\right\}^2 dz^2 - e^{-2mx}\left\{D_4 M^{\frac{1}{4}} t^n \exp[\frac{X_4 t^{1-4n}}{M(1-4n)}]\right\}^2 du^2 \tag{4.1.6}$$

Physical Quantities of power law model

The directional Hubble parameter are

$$H_x = \frac{X_1}{Mt^{4n}} + \frac{n}{t}, H_y = \frac{X_2}{Mt^{4n}} + \frac{n}{t}, H_z = \frac{X_3}{Mt^{4n}} + \frac{n}{t}, H_u = \frac{X_4}{Mt^{4n}} + \frac{n}{t}$$

The mean Hubble parameter H is

$$H = \frac{n}{t} \tag{4.1.7}$$

The Hubble parameter decreases as time t increases which shows e

The Average anisotropic expansion parameter A_m, shear scalar σ^2 and expansion scalar θ is

$$A_m = \frac{X^2}{4n^2 m^2 t^{2(4n-1)}}, \; X^2 = X_1^2 + X_2^2 + X_3^2 + X_4^2 \tag{4.1.8}$$

$$\sigma^2 = \frac{X^2}{2M^2t^{8n}} \tag{4.1.9}$$

$$\theta = \frac{4n}{t} \tag{4.1.10}$$

The value of q i.e. deceleration parameter is

$$q = \frac{1}{n} - 1 \tag{4.1.11}$$

The average scale factor R is $R = M^{\frac{1}{4}}t^n$ (4.1.12)

Pressure p_{WDF} of WDF model is

$$p_{WDF} = \frac{\gamma}{(8\mu\gamma - 8\pi - \mu)}\left[\left(\frac{2X_1^2}{M^2} + \frac{2X_2^2}{M^2} + \frac{X_3^2}{M^2} + \frac{X_4^2}{M^2} + \frac{X_1X_2}{M^2} - \frac{X_3X_4}{M^2}\right)t^{-8n} + 6n(n-1)t^{-2} + 8\mu\gamma\rho^*\right] - \gamma\rho^* \tag{4.1.13}$$

Energy density ρ_{WDF} of WDF model is

$$\rho_{WDF} = \frac{1}{(8\mu\gamma - 8\pi - \mu)}\left[\left(\frac{2X_1^2}{M^2} + \frac{2X_2^2}{M^2} + \frac{X_3^2}{M^2} + \frac{X_4^2}{M^2} + \frac{X_1X_2}{M^2} - \frac{X_3X_4}{M^2}\right)t^{-8n} + 6n(n-1)t^{-2} + 8\mu\gamma\rho^*\right] \tag{4.1.14}$$

By using $1 + z = \frac{R_0}{R}$ where R is scale factor (Piattella, 2018), the above physical parameters in terms of redshift z.

The pressure in terms of redshift z is

$$p_{WDF} = \frac{\gamma}{(8\mu\gamma - 8\pi - \mu)}\left[(2X_1^2 + 2X_2^2 + X_3^2 + X_4^2 + X_1X_2 - X_3X_4)(1+z)^8 + 6n(n-1)M^{1/2n}(1+z)^{2/n} + 8\mu\gamma\rho^*\right] - \gamma\rho^* \tag{4.1.15}$$

The energy density in terms of redshift z is

$$\rho_{WDF} = \frac{1}{(8\mu\gamma - 8\pi - \mu)}\left[(2X_1^2 + 2X_2^2 + X_3^2 + X_4^2 + X_1X_2 - X_3X_4)(1+z)^8 + 6n(n-1)M^{1/2n}(1+z)^{2/n} + 8\mu\gamma\rho^*\right] \tag{4.1.16}$$

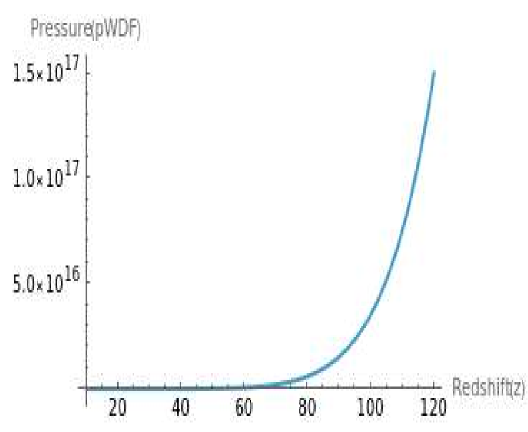

Fig.1

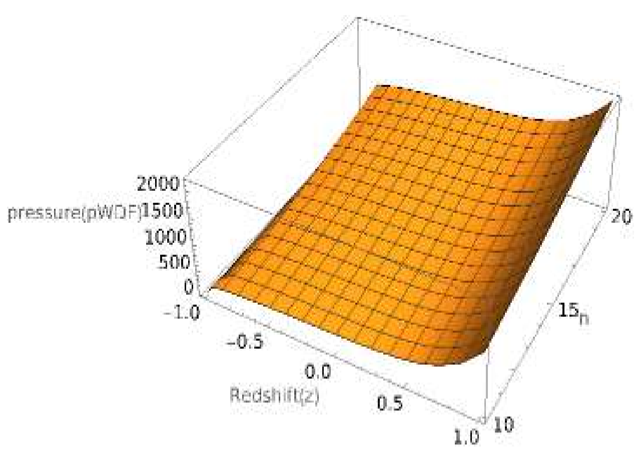

Fig.2

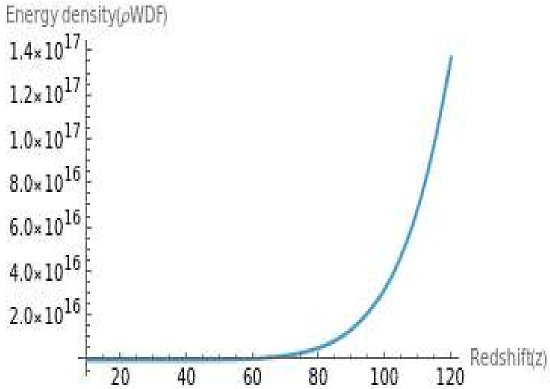

Fig.3

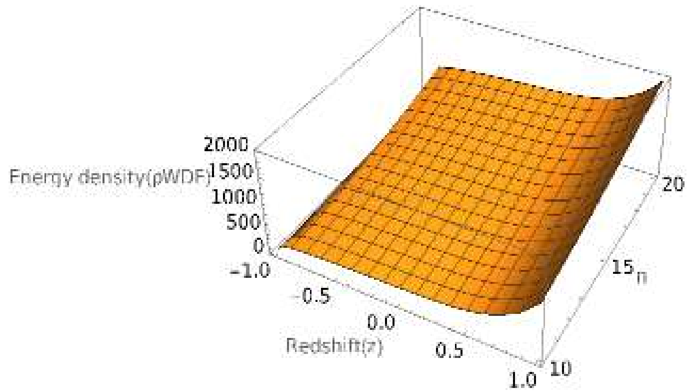

Fig.4

By choosing appropriate constant, Fig.1 and Fig. 2 Indicates variation of WDF pressure (p_{WDF}) against redshift (z) in 2D and 3D respectively. Also Fig.3 and Fig. 4 Indicates variation of WDF energy density (ρ_{WDF}) against redshift (z) in 2D and 3D respectively.

The mean Hubble parameter H in terms of redshift is

$$H = nM^{1/4n}(1+z)^{1/n} \tag{4.1.17}$$

The Average anisotropic expansion parameter A_m, shear scalar σ^2 and expansion scalar θ in terms of redshift z are

$$A_m = \frac{X^2}{4n^2m^2M^{-2+\frac{1}{2n}}(1+z)^{-8+\frac{2}{n}}},\ X^2 = X_1^2 + X_2^2 + X_3^2 + X_4^2 \tag{4.1.18}$$

$$\sigma^2 = \frac{X^2(1+z)^8}{2} \quad \text{and} \quad \theta = 4nM^{1/4n}(1+z)^{1/n} \tag{4.1.19}$$

Energy Conditions of the Model

The energy conditions plays a vital role in understanding the physical conditions and gravitational behavior of cosmological models. For the considered model, we discussed the standard energy conditions: the Weak Energy Condition (WEC), Null Energy Condition (NEC), Dominant Energy Condition (DEC), and Strong Energy Condition (SEC).

(i) Weak Energy Condition (WEC)

The WEC satisfy if $\rho_{WDF} \geq 0, \rho_{WDF} + p_{WDF} \geq 0.$

For the considered model, WEC satisfy if

$$\frac{1}{(8\mu\gamma-8\pi-\mu)}\left[\left(\frac{2X_1^2}{M^2}+\frac{2X_2^2}{M^2}+\frac{X_3^2}{M^2}+\frac{X_4^2}{M^2}+\frac{X_1X_2}{M^2}-\frac{X_3X_4}{M^2}\right)t^{-8n}+6n(n-1)t^{-2}+8\mu\gamma\rho^*\right] \geq 0 \text{ and}$$

$$\frac{1}{(8\mu\gamma-8\pi-\mu)}\left[\left(\frac{2X_1^2}{M^2}+\frac{2X_2^2}{M^2}+\frac{X_3^2}{M^2}+\frac{X_4^2}{M^2}+\frac{X_1X_2}{M^2}-\frac{X_3X_4}{M^2}\right)t^{-8n}+6n(n-1)t^{-2}+8\mu\gamma\rho^*\right] \geq \frac{\gamma}{(1+\gamma)}\rho^*.$$

(ii) Null Energy Condition (NEC)

The NEC satisfy if $\rho_{WDF} + p_{WDF} \geq 0$.

Which is same as the second condition of the WEC.

(iii) Dominant Energy Condition (DEC)

The DEC satisfy if $\rho_{WDF} \geq 0, \rho_{WDF} - |p_{WDF}| \geq 0$.

This condition gives two possibilities,

a) $\rho_{WDF} \geq p_{WDF} = \gamma(\rho_{WDF} - \rho^*)$ if $\rho_{WDF} \geq \rho^*$.
b) $\rho_{WDF} \geq p_{WDF} = \gamma(\rho_{WDF} - \rho^*)$ if $\rho_{WDF} < \rho^*$.

For both the cases $\rho_{WDF} \geq \frac{\gamma}{(1+\gamma)}\rho^*$.

For our model,

$$\frac{1}{(8\mu\gamma-8\pi-\mu)}\left[\left(\frac{2X_1^2}{M^2}+\frac{2X_2^2}{M^2}+\frac{X_3^2}{M^2}+\frac{X_4^2}{M^2}+\frac{X_1X_2}{M^2}-\frac{X_3X_4}{M^2}\right)t^{-8n}+6n(n-1)t^{-2}+8\mu\gamma\rho^*\right] \geq \frac{\gamma}{(1+\gamma)}\rho^*.$$

Which also coincides with the weak energy condition.

(iv) Strong Energy Condition (SEC)

The SEC satisfy if $\rho_{WDF} + 3p_{WDF} \geq 0$.

For the considered model, SEC satisfy if

$$\frac{1}{(8\mu\gamma-8\pi-\mu)}\left[\left(\frac{2X_1^2}{M^2}+\frac{2X_2^2}{M^2}+\frac{X_3^2}{M^2}+\frac{X_4^2}{M^2}+\frac{X_1X_2}{M^2}-\frac{X_3X_4}{M^2}\right)t^{-8n}+6n(n-1)t^{-2}+8\mu\gamma\rho^*\right] \geq \frac{3\gamma}{(1+3\gamma)}\rho^*.$$

At late time i.e. $t \to \infty$, the ρ_{WDF} decreases. Hence the above inequality is not satisfied, which indicates violation of SEC. The satisfaction of WEC, NEC, and DEC confirms the physical acceptability of the WDF model under suitable bounds on model parameters. The violation of the SEC represents the accelerated expansion of the Universe, as observed in current cosmological data. This supports the interpretation of the WDF as a viable candidate for dark energy.

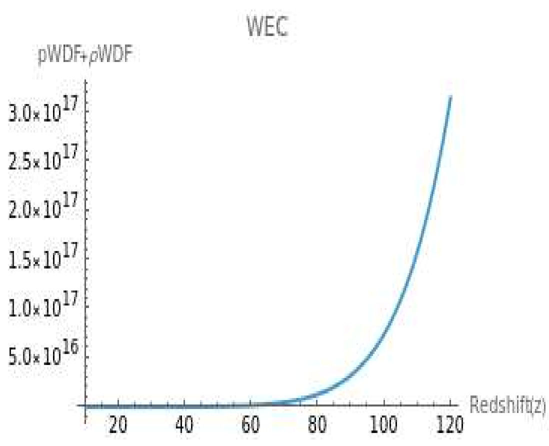

Fig. 5

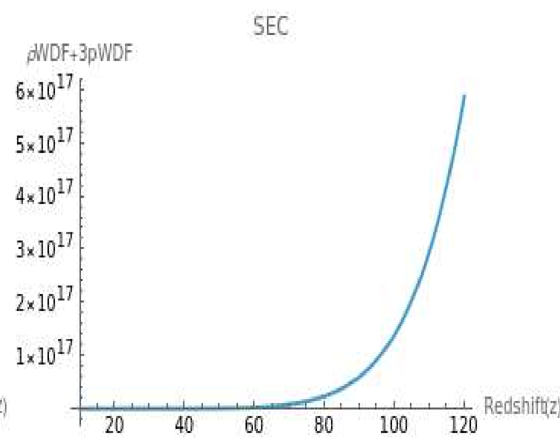

Fig. 6

Fig. 5 and Fig.6 shows the graphical representation of WEC and SEC respectively with respect to redshift z.

Stability of the model

The stability of the model is examined through the analysis of the ratio of sound speed given by $\frac{dp}{d\rho} = C_s^2$. The model is considered stable when $C_s^2 > 0, i.e. \frac{dp}{d\rho} > 0$, indicates that the pressure increases with energy density. Also the model becomes unstable when $C_s^2 < 0, i.e. \frac{dp}{d\rho} < 0$, which indicates a negative response of pressure to change in energy density, investigated by (Nimkar & Hadole, 2023) and (Agrawal & Nile, 2024).

For the present model we obtained $\frac{dp_{WDF}}{d\rho_{WDF}} = \gamma$

Here the chosen range of the parameter γ is $0 < \gamma < 1$. Hence the model is stable within the range $0 < \gamma < 1$.

Cosmographic parameters

In this section, we consider the measurement of cosmological distances at significant redshifts, where the impact of the cosmological expansion on the computation of the distance can no longer be neglected. Such measurements help us to find out the expansion history of the universe and for determining whether the expansion is accelerating or decelerating, along with the associated rate.

The statefinder parameter

The new pair of parameters is proposed by (Sahni, et al., 2003) and (Alam , et al., 2003) called as the statefinder parameters $\{r, s\}$. The statefinder pair is defined as

$$r = q + 2q^2 - \frac{\dot{q}}{H} = \frac{\dddot{R}}{RH^3} \quad \text{and} \quad s = \frac{r-1}{3(q-\frac{1}{2})}.$$

By using statefinder parameters we can examine the expansion history of the universe through higher derivatives of the expansion factor. (Chang, et al., 2008) studied statefinder parameter for five dimensional cosmology and (Popov, 2011), (Shaikh, et al., 2021), (Solanki & Sahoo, 2022) also studied statefinder parameter. Different values of statefinder parameter pair $\{r, s\}$ exhibit different dark energy model, i.e. for ΛCDM model, $r = 1, s = 0$, for SCDM model, $r = 1, s = 1$, for HDE model, $r = 1, s = \frac{2}{3}$, for Quintessence model, $r < 1, s > 0$, for CG model $r > 1, s < 0$.

For our model (4.1.6), the statefinder parameters pair $\{r, s\}$ is obtained as

$$r = \frac{n^2-3n+2}{n^2} \quad \text{and} \quad s = \frac{2}{3n}.$$

The relation between the statefinder parameter r and s is $r = \frac{9s^2-9s+2}{2}$.

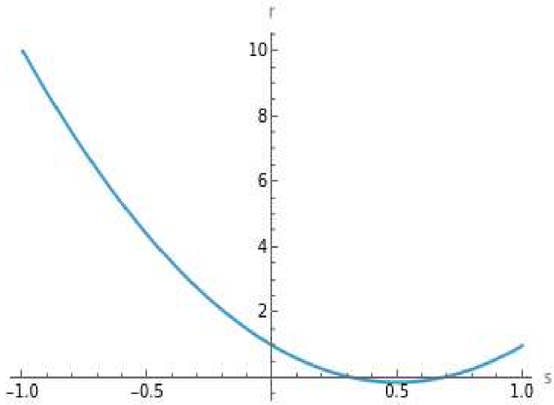

Fig. 7

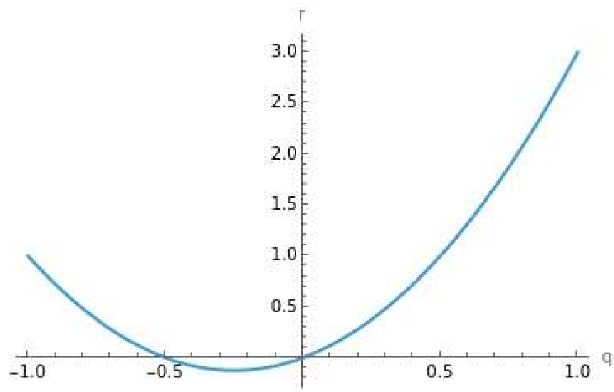

Fig. 8

The graphical representation of r with respect to s is shown in fig. 7 and fig. 8 is graphical representation of r with respect to q. In r-s graph initially graph shows Chaplygian gas model $(r > 1, s < 0)$ then graph shows ΛCDM model $(r = 1, s = 0)$.

Look-back time

The look-back time $t_L(z)$ is defined as the elapsed time between the present age of the universe t_0 and the time t when the light from a cosmic source at a particular red-shift z was emitted. The Look-back time t_L (Krisciunas, 1993) is defined as $t_L(z) = t_0 - t(z) = \int_R^{R_0} \frac{dR}{\dot{R}}$ where R_0 is the present value of the scale factor and $1 + z = \frac{R_0}{R}$ also t_0 is current age of the universe and z is the redshift of the measured amount of light from the galaxy.

The Look-back time in the redshift for the studied model is obtained as

$$t_L(z) = \frac{1}{M^{1/4n}}\left[1 - \frac{1}{(1+z)^{1/n}}\right]$$

Proper distance $\mathbf{d_p(z)}$

The proper distance $d_p(z)$ is defined as the distance between a cosmic source-emitting light at any instant $t = t_1$ located at $r = r_1$ with redshift z an observer at $r = 0$ and $t = t_0$ receiving the light from the source emitted, that is $d_p(z) = r_1 R_0$ where $r_1 = \int_0^z \frac{dz'}{H(z')}$.

Hence the proper distance $d_p(z)$ (Hogg, 1999) is defined as $d_p(z) = R_0 \int_0^z \frac{dz'}{H(z')}$

For the studied model, the proper distance is obtained as

$$d_p(z) = \begin{cases} \frac{1}{nM^{1/4n}} \frac{(1-z)^{1-\frac{1}{n}} - 1}{1-\frac{1}{n}}, & for\ n \neq 1 \\ \frac{1}{nM^{1/4n}} \ln(1+z), & for\ n = 1 \end{cases}$$

Luminosity distance $\mathbf{d_L(z)}$

The luminosity distance $d_L(z)$ is a fundamental concept in observational cosmology which connects an objects intrinsic luminosity to its observed flux. In an expanding universe, for a light source at redshift z, the luminosity distance is related to the proper distance. The relation between Luminosity distance and Proper distance is

$$d_L(z) = (1+z)d_p(z)$$

$$d_L(z) = \begin{cases} \frac{1+z}{nM^{1/4n}} \frac{(1-z)^{1-\frac{1}{n}-1}}{1-\frac{1}{n}}, & for\ n \neq 1 \\ \frac{1+z}{nM^{1/4n}} \ln(1+z), & for\ n = 1 \end{cases}$$

Angular diameter distance $d_A(z)$

The angular diameter distance is a cosmological distance measure that relates the physical size of an object to its angular size it appears to have on the sky. The angular diameter distance is defined as $d_A(z) = \frac{d_L(z)}{(1+z)^2}$ (Demianski, et al., 2003).

The angular diameter distance is obtained for (4.1.6) model is

$$d_L(z) = \begin{cases} \frac{1}{nM^{1/4n}} \frac{(1-z)^{1-\frac{1}{n}-1}}{(1-\frac{1}{n})(1+z)}, & for\ n \neq 1 \\ \frac{1}{nM^{1/4n}} \ln(1+z), & for\ n = 1 \end{cases}$$

Jerk parameter j

Jerk parameter measures the rate of change of the acceleration of the universe and is defined as a dimensionless third derivative of the scale factor with respect to the cosmic time t (Visser, 2004).

$$j = \frac{1}{H^3 R} \frac{d^3 R(t)}{dt^3}$$

$$j = \frac{(n-1)(n-2)}{n^2}$$

Cosmic snap parameter s

The cosmic snap parameter in cosmology is dimensionless parameter and it is useful in describing deviations from ΛCDM or kinematic cosmology. The snap parameter is defined as the dimensionless fourth derivative of the scale factor with respect to cosmic time t (Visser, 2004).

$$s = \frac{1}{H^4 R} \frac{d^4 R(t)}{dt^4}$$

$$s = \frac{(n-1)(n-2)(n-3)}{n^3}$$

4.2 Exponential expansion model

$V = Le^{4Nt}$, where L, N are constant.

$$P_1(t) = D_1 L^{\frac{1}{4}} e^{Nt} \exp[\frac{-X_1 e^{-4nt}}{4NL}] \tag{4.2.1}$$

$$P_2(t) = D_2 L^{\frac{1}{4}} e^{Nt} \exp[\frac{-X_2 e^{-4nt}}{4N}] \tag{4.2.2}$$

$$P_3(t) = D_3 L^{\frac{1}{4}} e^{Nt} \exp[\frac{-X_3 e^{-4n}}{4N}] \tag{4.2.3}$$

$$P_4(t) = D_4 L^{\frac{1}{4}} e^{Nt} \exp[\frac{-X_4 e^{-4nt}}{4N}] \tag{4.2.4}$$

where $D_1 D_2 D_3 D_4 = 1$ and $X_1 + X_2 + X_3 + X_4 = 0$.

Then the solution is

$$ds^2 = dt^2 - \left\{D_1 L^{\frac{1}{4}} e^{Nt} \exp\left[\frac{-X_1 e^{-4n}}{4NL}\right]\right\}^2 dx^2 - e^{-2mx} \left\{D_2 L^{\frac{1}{4}} e^{Nt} \exp[\frac{-X_2 e^{-4nt}}{4N}]\right\}^2 dy^2 - e^{-2mx} \left\{D_3 L^{\frac{1}{4}} e^{Nt} \exp[\frac{-X_3 e^{-4nt}}{4NL}]\right\}^2 dz^2 - e^{-2mx} \left\{D_4 L^{\frac{1}{4}} e^{Nt} \exp[\frac{-X_4 e^{-4nt}}{4NL}]\right\}^2 du^2 \tag{4.2.5}$$

Physical Parameters of Exponential Expansion model

The directional Hubble parameter are

$$H_x = \frac{X_1}{Le^{4nt}} + N, H_y = \frac{X_2}{Le^{4nt}} + N, H_z = \frac{X_3}{Le^{4nt}} + N, H_u = \frac{X_4}{Le^{4nt}} + N$$

Mean Hubble parameter H is obtained as

$$H = N \tag{4.2.6}$$

Here we get the constant Hubble parameter.

The Average anisotropic expansion parameter A_m, shear scalar σ^2 and expansion scalar θ is

$$A_m = \frac{X^2}{4N^2 L^2 (e^{4N})^2}, \quad X^2 = X_1^2 + X_2^2 + X_3^2 + X_4^2 \tag{4.2.7}$$

$$\sigma^2 = \frac{X^2}{2L^2 e^{8N}} \tag{4.2.8}$$

$$\theta = 4N \tag{4.2.9}$$

The value of q i.e. Deceleration parameter is

$$q = -1 \tag{4.2.10}$$

The pressure p_{WDF} of the WDF is

$$p_{WDF} = \frac{\gamma}{(8\mu\gamma - 8\pi - \mu)} \left[\left(\frac{2X_1^2}{L^2} + \frac{2X_2^2}{L^2} + \frac{X_3^2}{L^2} + \frac{X_4^2}{L^2} + \frac{X_1 X_2}{L^2} - \frac{X_3 X_4}{L^2}\right) e^{-8Nt} + 6N^2 + 8\mu\gamma\rho^*\right] - \gamma\rho^* \tag{4.2.11}$$

The energy density ρ_{WDF} of the WDF is

$$\rho_{WDF} = \frac{1}{(8\mu\gamma - 8\pi - \mu)} \left[\left(\frac{2X_1^2}{L^2} + \frac{2X_2^2}{L^2} + \frac{X_3^2}{L^2} + \frac{X_4^2}{L^2} + \frac{X_1 X_2}{L^2} - \frac{X_3 X_4}{L^2}\right) e^{-8Nt} + 6N^2 + 8\mu\gamma\rho^*\right] \tag{4.2.12}$$

By using $1 + z = \frac{R_0}{R}$ where R is scale factor, the above physical parameters in terms of redshift z.

The pressure in terms of redshift z is

$$p_{WDF} = \frac{\gamma}{(8\mu\gamma - 8\pi - \mu)} [(2X_1^2 + 2X_2^2 + X_3^2 + X_4^2 + X_1 X_2 - X_3 X_4)(1 + z)^8 + 6N^2 + 8\mu\gamma\rho^*] - \gamma\rho^* \tag{4.2.13}$$

The energy density in terms of redshift z is

$$\rho_{WDF} = \frac{1}{(8\mu\gamma-8\pi-\mu)}\left[(2X_1^2 + 2X_2^2 + X_3^2 + X_4^2 + X_1X_2 - X_3X_4)(1+z)^8 + 6N^2 + 8\mu\gamma\rho^*\right] \quad (4.2.14)$$

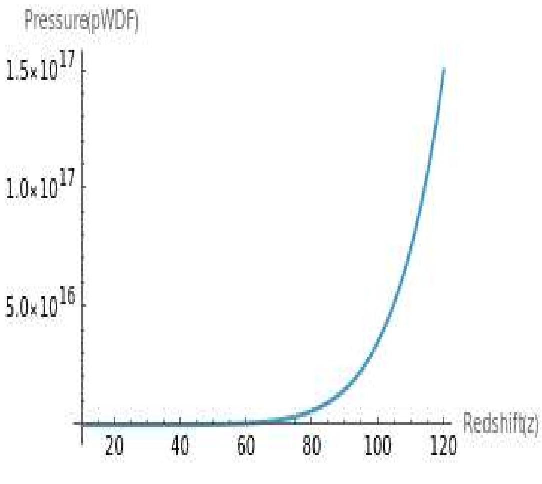

Fig. 9

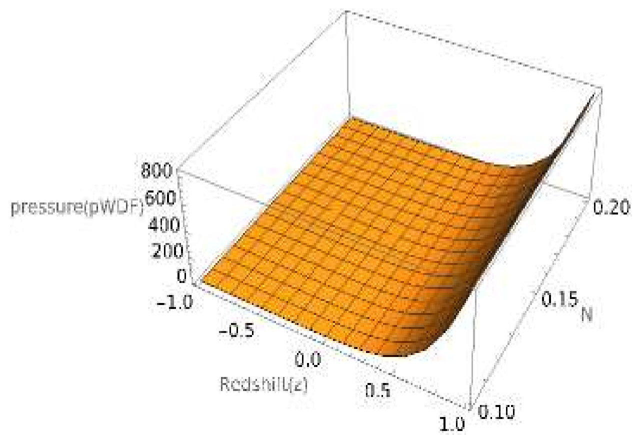

Fig. 10

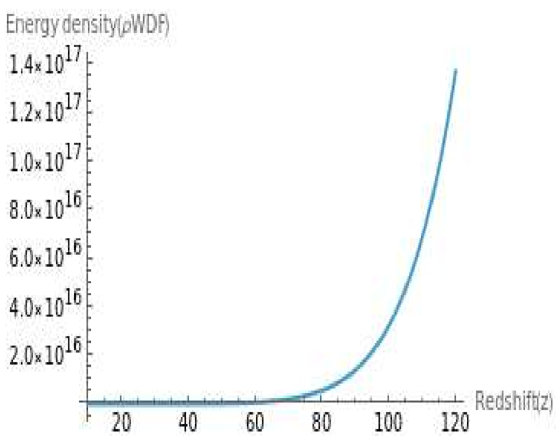

Fig. 11

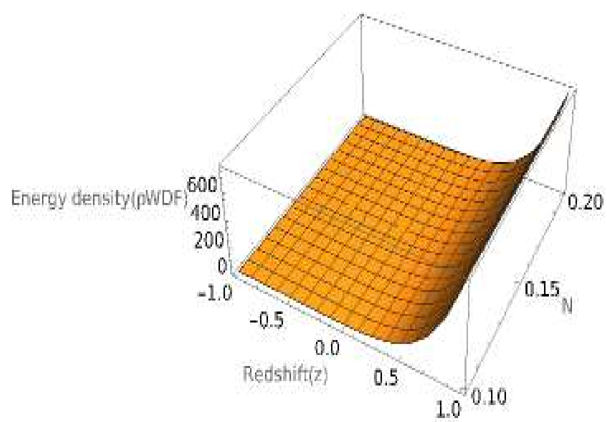

Fig. 12

By choosing appropriate constant, Fig.9 and Fig. 10 Indicates variation of WDF pressure (p_{WDF}) against redshift (z) in 2D and 3D respectively. Also Fig.11 and Fig. 12 Indicates variation of WDF energy density (ρ_{WDF}) against redshift (z) in 2D and 3D respectively.

Energy Conditions of the Model

(i) Weak Energy Condition (WEC)

The WEC satisfy if $\rho_{WDF} \geq 0, \rho_{WDF} + p_{WDF} \geq 0$.

For the considered model, WEC satisfy if

$$\frac{1}{(8\mu\gamma-8\pi-\mu)}\left[\left(\frac{2X_1^2}{L^2} + \frac{2X_2^2}{L^2} + \frac{X_3^2}{L^2} + \frac{X_4^2}{L^2} + \frac{X_1X_2}{L^2} - \frac{X_3X_4}{L^2}\right)e^{-8Nt} + 6N^2 + 8\mu\gamma\rho^*\right] \geq 0 \text{ and}$$

$$\frac{1}{(8\mu\gamma-8\pi-\mu)}\left[\left(\frac{2X_1^2}{L^2}+\frac{2X_2^2}{L^2}+\frac{X_3^2}{L^2}+\frac{X_4^2}{L^2}+\frac{X_1X_2}{L^2}-\frac{X_3X_4}{L^2}\right)e^{-8Nt}+6N^2+8\mu\gamma\rho^*\right]\geq\frac{\gamma}{(1+\gamma)}\rho^*.$$

(ii) Null Energy Condition (NEC)

The NEC satisfy if $\rho_{WDF}+p_{WDF}\geq 0$.

Which is same as the second condition of the WEC.

(iii) Dominant Energy Condition (DEC)

The DEC satisfy if $\rho_{WDF}\geq 0, \rho_{WDF}-|p_{WDF}|\geq 0$.

This condition gives two possibilities,

c) $\rho_{WDF}\geq p_{WDF}=\gamma(\rho_{WDF}-\rho^*)$ if $\rho_{WDF}\geq\rho^*$.
d) $\rho_{WDF}\geq p_{WDF}=\gamma(\rho_{WDF}-\rho^*)$ if $\rho_{WDF}<\rho^*$.

For both the cases $\rho_{WDF}\geq\frac{\gamma}{(1+\gamma)}\rho^*$.

For our model,

$$\frac{1}{(8\mu\gamma-8\pi-\mu)}\left[\left(\frac{2X_1^2}{L^2}+\frac{2X_2^2}{L^2}+\frac{X_3^2}{L^2}+\frac{X_4^2}{L^2}+\frac{X_1X_2}{L^2}-\frac{X_3X_4}{L^2}\right)e^{-8Nt}+6N^2+8\mu\gamma\rho^*\right]\geq\frac{\gamma}{(1+\gamma)}\rho^*.$$

Which also coincides with the weak energy condition.

(iv) Strong Energy Condition (SEC)

The SEC satisfy if $\rho_{WDF}+3p_{WDF}\geq 0$.

For the considered model, SEC satisfy if

$$\frac{1}{(8\mu\gamma-8\pi-\mu)}\left[\left(\frac{2X_1^2}{L^2}+\frac{2X_2^2}{L^2}+\frac{X_3^2}{L^2}+\frac{X_4^2}{L^2}+\frac{X_1X_2}{L^2}-\frac{X_3X_4}{L^2}\right)e^{-8Nt}+6N^2+8\mu\gamma\rho^*\right]\geq\frac{3\gamma}{(1+3\gamma)}\rho^*.$$

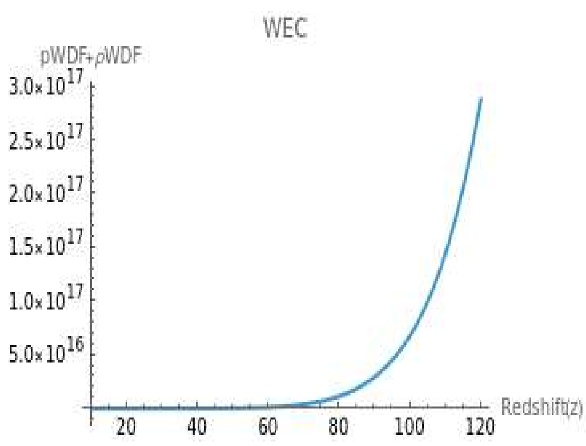

Fig. 13

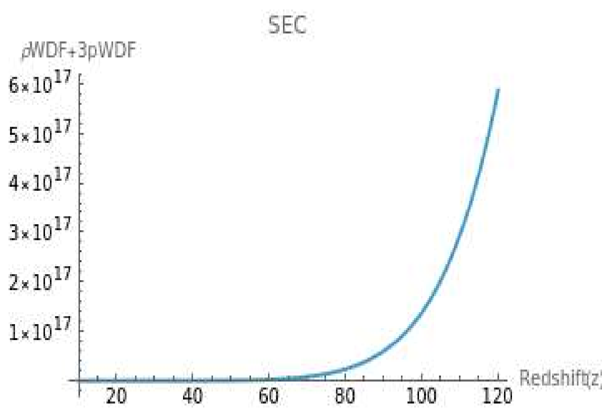

Fig. 14

Fig. 13 and Fig.14 shows the graphical representation of WEC and SEC respectively with respect to redshift z.

Stability of the model

For the present model we obtained

$$\frac{dp_{WDF}}{d\rho_{WDF}} = \gamma$$

Here the chosen range of the parameter γ is $0 < \gamma < 1$. Hence the model is stable within the range $0 < \gamma < 1$.

Cosmographic parameters

In this section, we consider the measurement of cosmological distances at significant redshifts, where the impact of the cosmological expansion on the computation of the distance can no longer be neglected. Such measurements help us to find out the expansion history of the universe and for determining whether the expansion is accelerating or decelerating, along with the associated rate.

Statefinder parameters

The statefinder parameters pair $\{r, s\}$ for the model (4.2.5), is obtained as

$r = 1$ and $s = 0$.

Hence for exponential expansion, our model represent ΛCDM model.

Look-back time $t_L(z)$

For the studied model (4.2.5), the Look-back time in the redshift is obtained as

$$t_L(z) = \frac{1}{N}\ln(1 + z)$$

Proper distance $d_p(z)$

The proper distance is obtained as $d_p(z) = \frac{z}{N}$.

In exponential expansion the proper distance increases linearly with redshift.

Luminosity distance $d_L(z)$

The Luminosity distance is obtained as $d_L(z) = \frac{z(1+z)}{N}$

Angular diameter distance $d_A(z)$

The angular diameter distance is obtained as $d_A(z) = \frac{z}{N(1+z)}$

Jerk parameter j

Jerk parameter for the model (4.2.5) is obtained as $j = 1$

Cosmic snap parameter s

The cosmic snap parameter in cosmology is obtained as $s = 1$

5. Concluding remarks

In this paper, we studied a bianchi type-V cosmological model filled with wet dark fluid (WDF) which is one of the form of dark energy in $f(R,T)$ theory of gravitation. The solutions of the field equation is obtained by considering two volumetric expansion laws model as i) power law expansion model and ii) exponential expansion law model.

Power law model

In power law model, the mean Hubble parameter decreases to zero as time $t \to \infty$ and increases as redshift $z \to \infty$. The expansion scalar, shear scalar and anisotropic expansion parameter gradually decreases to zero as time t increases. Hence we can say that the universe is expanding with small rate of expansion and also universe starts with anisotropic state but can approach to isotropy. The deceleration parameter of the model is depend on the value of n, if $n = 1$, the deceleration parameter is zero which conclude that the each galaxy moves with constant speed, if $n > 1$, the value of deceleration parameter is negative which conclude that the universe is accelerating so the range of q is $-1 < q < 0$. The wet dark fluid pressure $p_{WDF} \to 0$ as $t \to \infty$, i. e. pressureless dark energy (Chan, 2015). The wet dark fluid density ρ_{WDF} decreases with time t increases which concludes expansion of the universe. Also p_{WDF} and ρ_{WDF} increases as the redshift z increases which indicates early universe was higher in temperature and small in size. The above observation are in good agreement with the current observations. The statefinder parameter pair $\{r, s\} \to \{1,0\}$ as $n \to \infty$ hence our model approaches to ΛCDM model for large value of n. The solutions of observational parameters such as look-back time, proper distance angular diameter and snap parameter are in good agreement with the current observations. The value of the jerk parameter j approaches to 1 for large value of n which is consistent with the flat ΛCDM model. The derived model is stable within the range $0 < \gamma < 1$. The model satisfy the standard energy conditions WEC, NEC, and DEC confirms the physical acceptability of the WDF model under suitable bounds on model parameters. The violation of the SEC represents the accelerated expansion of the Universe, as observed in current cosmological data.

Exponential law model

In exponential expansion law model, the values of mean Hubble parameter and expansion scalar are constant. The shear scalar and anisotropic expansion parameter gradually decreases to zero as time t increases. Hence we can say that the universe is expanding with small rate of expansion and also universe starts with anisotropic state but can approach to isotropy. The value of deceleration parameter is $q = -1$ which is constant deceleration parameter and indicates that the universe is accelerating. The wet dark fluid pressure p_{WDF} and wet dark fluid density ρ_{WDF} increases as the redshift z increases which indicates early universe was higher in temperature and small in size. The statefinder parameter pair $\{r, s\} = \{1,0\}$ hence our model represents the ΛCDM model. The solutions of observational parameters such as look-back time, proper distance and angular diameter are in good agreement with the current observations. The value of the jerk parameter is $j = 1$ which is consistent with the flat ΛCDM model. The derived model is stable within the range $0 < \gamma < 1$. The above observation are in good agreement with the current observations.

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