A SHORT NOTE ON FRAME SCALING IN FINITE DIMENSIONAL HILBERT SPACES

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Abstract. In this short note, frames in Hilbert spaces are discussed and studied. Frame scaling of frames in finite dimensional Hilbert spaces is considered and few results regarding frame scaling in Hilbert spaces are obtained. Dual of frames are considered and it is proved that a frame is scalable if and only if its dual frame is scalable. Union of two frames is considered and it is proved that if two frames are scalable, then the union is also a scalable frame. Illustrative examples are provided in the note.

1. Introduction

Frames and bases in Hilbert spaces and Banach spaces play a vital role in many applications. Frames were introduced long ago in 1952 by Duffin and Schaeffer as a generalization of bases in Hilbert spaces. They introduced frames to use as a vital tool in the study non-harmonic Fourier series. Duffin and Schaeffer introduced frames for a particular Hilbert spaces of the form $L^2[a, b]$ i.e. the space of square integrable function over the interval $[a, b]$. Later on, the notion of frames was extended to other Hilbert spaces.

After more than thirty five years, in 1986, Daubechies, Grossmann and Meyer, while studying frames in the space of square integrable functions over the set of real numbers, observed and noticed that the elements of a frame can be used to approximate any function of the space $L^2(\mathbb{R})$ in a suitable way. In particular, they observed that a function in $L^2(\mathbb{R})$ can be represented by a series in terms of elements of a frame and the series representation is very much similar to the series representation given by a basis, like orthonormal basis. Therefore, frames can be considered as one of the generalizations of orthonormal bases or simply bases in Hilbert spaces. Since the representations of vectors by frames are not unique, the redundant expansions given by frames are more useful and advantageous over the representations given by bases in a variety of practical applications in many fields. The advantage of redundancy of frames has been exploited in many applications since then.

Now a days, the concept of frames is regarded as an important and integral tool in study of various areas of theory and applications; like representation of signals, characterization of many

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function spaces and other related fields of applications such as signal processing and image processing, filter bank theory, wireless communications and sigma-delta quantizations. The list of fields where the theory of frames is used in theory or applications has been expanding over the years. Probably, the theory of frames interacts with most of active research areas of mathematics and electronics & communication. For more literature on the frame theory and related concepts, readers may refer to papers and book in references.

To deal with some specific requirements in applications, a lot of new generalizations of frames are being introduced and studied by various researchers over the last 20 years. These generalizations are introduced for more applications of frames in different fields. Some of these are for specific uses in the theory besides catering to applications.

Frames have been playing an active and significant role in many areas of mathematics and sciences, particularly the theory of signal processing and image processing, but today researchers have found applications frames to packet based network communications, wireless sensor networks, distributed processing, quantum information theory, bio-medical engineering , compressed sensing, fingerprinting, spectral theory, and many more such areas of science and engineering.

Most of applications of frames arise from the ability to deliver redundant (not unique), yet stable expansions or representations of vectors of a space by the elements of a frame for the space. The redundancy of a frame is typically characterized and utilized in applications which require robustness of the frame coefficients to noise, erasures, quantization, truncation etc. In such settings, tight frames are more suitable where tight frames can provide faster convergence and thus faster recovery of a vector. It is known that unit norm tight frames are characterized in terms of the frame potential. Thus tight frames are connected with frame potential, the notion which is widely used and studied in physical sciences. There have been various works on constructions of tight frames by various researchers.Different researchers have provided different schemes with which a tight frame can be produced. However, it is favourable and desirable to construct tight frames by just scaling each frame vector as it is non-invasive and frame properties such as erasure resilience or sparse expansions are left undisturbed by such modifications. Thus using a set of suitable scalars, it is convenient and worthy to produce a tight frame from a given frame, if possible.

To generate a tight frame using a given frame, a process of scaling of frames is used by researchers recently. In scaling, the vectors are scaled i.e. dilated or contracted which will not only generate a tight frame but also it will not disturb or distort the basic properties of the frame. Hence, in view of applications, the nature of the frame doesn't undergo any fundamental change.

To make readers aware of such an important process of generating tight frames, the present note is prepared. In the note, a study of frame scaling in Hilbert spaces is considered and discussed. Few results are obtained in this direction. The discussion is initiated and supported with relevant examples. The results are also supported by examples for better understanding of the result obtained.

The paper is organized as follows: The first section contains the introduction of the paper. In this section, frames, tight frames and application of frames are given with the relevant literature for proper understanding. In Section 2, we give notations and terminology used in the context of fames in finite and infinite dimensional Hilbert spaces together with some basic results related to frames in Hilbert spaces. This section is also devoted to the concept of frames in finite dimensional Hilbert spaces. The basic definitions, examples and results related to frames or frame scaling are listed in this section. Some supportive examples are provided with the results in the section. In Section 3, frames and frame scaling in finite dimensional Hilbert spaces are considered and studied. A few concerned results are obtained. We conclude the paper with a formal Conclusion after Section 3. References are provided at the end. The list of references is not exclusive and therefore readers may find some other interesting and useful references related to specific applications of frames.

2. Preliminaries

Before proceeding further, we require some definitions and terminology which will be used in the paper.

Throughout the paper, $\mathcal{H}_N$ denoted a N-dimensional Hilbert space for some finite $N \in \mathbb{N}$, $\mathcal{H}$ denotes an infinite dimensional separable Hilbert space, I or $\mathbb{N}$ denote a countable infinite index set and $M = \{1, 2, \cdots, m\}$ denotes a finite index set. The cardinality of a set D is denoted by $|D|$. The span closure of x_i is denoted by $[x_i]$.

Duffin and Schaeffer introduced the notion of frame for Hilbert spaces by abstracting the notion given by Gabor. The formal definition of a frame in a Hilbert space is given as below.

Definition (Frame). A family of vectors or elements $\Phi = \{\varphi_i\}_{i\in I}$ is said to be a frame for a separable Hilbert space $\mathcal{H}$ if there exist constants $0 < A \le B < \infty$ such that for all $x \in \mathcal{H}$,

$$A\|x\|^2 \le \sum_{i\in I} |\langle x, \varphi_i\rangle|^2 \le B\|x\|^2.$$

The corresponding definition for finite dimensional Hilbert spaces is

A family of M vectors or elements $\Phi = \{\varphi_i\}_{i=1}^{M}$ is said to be a frame for a Hilbert space $\mathcal{H}_N$ if there exist constants $0 < A \le B < \infty$ such that for all $x \in \mathcal{H}_N$,

$$A\|x\|^2 \le \sum_{i=1}^{M} |\langle x, \varphi_i\rangle|^2 \le B\|x\|^2.$$

The above inequality is referred as frame inequality. If the inequality of a frame is satisfied, then the upper half of the inequality is also satisfied by any other constant bigger than B and the lower half of the inequality is also satisfied by any other constant smaller than A. Then one may look for the largest such A and the smallest such B. The largest of all constants A and smallest of all constants B satisfying the above inequalities are referred as the optimal lower and the optimal upper frame bounds, respectively or simply the lower and the upper frame bounds, respectively. When $A = B$, the frame is called an A-tight frame.

So in case of an A-tight frame, the frame inequality will reduces to equality

$$A\|x\|^2 = \sum_{i\in I} |\langle x, \varphi_i\rangle|^2$$

in infinite case or in finite case, it is

$$A\|x\|^2 = \sum_{i=1}^{M} |\langle x, \varphi_i\rangle|^2 .$$

Furthermore, a frame is a normalized tight or a Parseval frame if $A = B = 1$. A frame is said to be an exact frame if removal of any element from the collection leaves the collection no longer a frame for the space. If a frame can be made exact by removing or dropping finite number of elements from the collection, then the frame is referred as a near-exact frame. In case of $\|\varphi_i\|= 1$, for all i, then the frame is called a unit norm frame.

The concept of finite unit norm tight frame is recently studied by researchers extensively. These frames are useful in description of the symmetry transformations, in symmetric form of invariants, in defining new mathematical objects with physical meaning and in optimal packings.

Example 1. Let $\mathcal{H} = l_2(\mathbb{N})$, the space of square summable sequences over $\mathbb{N}$ and let $\{e_i\}_{i\in\mathbb{N}}$ be the orthonormal basis. Let $\Phi = \{\varphi_i = e_i\}_{i\in\mathbb{N}}$. Then

$$\|x\|^2= \sum_{i\in I} |\langle x, \varphi_i\rangle|^2 .$$

So, taking $A = B = 1$, Φ is an exact unit norm tight frame for $\mathcal{H}$.

Example 2. Let $\mathcal{H} = l_2(\mathbb{N})$, the space of square summable sequences over $\mathbb{N}$ and let $\{e_i\}_{i\in\mathbb{N}}$ be the orthonormal basis. Let $\Phi = \left\{\varphi_1 = \frac{e_1}{1}, \varphi_2 = \frac{e_2}{2}, \varphi_3 = \frac{e_3}{3}, \cdots, \varphi_i = \frac{e_i}{i}, \cdots\right\}$. Then

$$\sum_{i\in I} |\langle x, \varphi_i\rangle|^2 \le \|x\|^2.$$

But, there does not exist any positive constant A with which the lower half of the inequality is satisfied. Thus Φ is not a frame for $\mathcal{H}$.

A collection or a sequence Φ which satisfies the upper half of the frame inequality is called a Bessel sequence. A frame is a Bessel sequence but the converse it not true as one can notice in Example 2.

Example 3. Let $\mathcal{H} = \mathbb{R}^3(\mathbb{R})$, the 3-dimensional space.
Let $\Phi = \left\{\varphi_1 = (1, 0, 0), \varphi_2 = (0, \frac{1}{2}, 0), \varphi_3 = (0, 0, 1)\right\}$. Then

$$\frac{1}{2}\|x\|^2\le \sum_{i=1}^{3} |\langle x, \varphi_i\rangle|^2 \le \|x\|^2.$$

So, taking $A = \frac{1}{2}, B = 1$, Φ is an exact frame for $\mathcal{H}$. But it is not tight and not unit norm frame.

Example 4. Let $\mathcal{H} = \mathbb{R}^3(\mathbb{R})$, the 3-dimensional space.
Let $\Phi = \left\{\varphi_1 = (\frac{1}{2}, 0, 0), \varphi_2 = (\frac{1}{2}, 0, 0), \varphi_3 = (0, 1, 0), \varphi_4 = (0, 0, 1)\right\}$. Then

$$\|x\|^2\le \sum_{i=1}^{4} |\langle x, \varphi_i\rangle|^2 \le \|x\|^2.$$

So, taking $A = B = 1$, Φ is a tight frame for $\mathcal{H}$. But it is not exact and not unit norm frame.

Similarly, one can construct other types of frames in infinite dimensional separable Hilbert spaces and finite dimensional Hilbert spaces. One can note that an exact frame in a finite dimensional Hilbert space has the number of elements in the collection equal to the dimension of the space (Example 3) and the frame is inexact if it has more number of elements in the collection than the dimension of the space (Example 4).

If someone considers a matrix of dimension $N \times M$ with the i^{th} column as the vector φ_i, then the frame Φ can be expressed in form of a matrix. In the same sense, the set of frames with M vectors in $\mathcal{H}_N$ will be denoted by $\mathcal{F}(M, N)$.

Given a frame, an operator, called frame operator, exists which gives representations of vectors in terms of elements of the frame. The frame operator is denoted by S.

Regarding tight frames, the following result is well known.

Theorem 1 A frame Φ with the frame operator S is A-tight if and only if

$$S := \Phi\Phi^* = \sum_{i=1}^{M} \varphi_i \varphi_i^* = AI_N,$$

where I_N is the identity matrix in $\mathcal{H}_N$. Some other related results with proofs can be found in the references.

As discussed in the introduction, it is desirable in applications to construct suitable tight frames by just scaling each frame vector with the help of scalars. Scaling of frame vectors is convenient as it is non-invasive and some frame properties such as erasure resilience or sparse expansions are left untouched or undisturbed by scaling.

The process of generating tight frames from given frames or sequences is referred as frame scaling and the frames which can be converted into tight frames by scaling are called scalable frames.

The scalable frames in finite dimensional Hilbert spaces are defined as

Definition (Scalable Frame). A frame $\Phi = \{\varphi_i\}_{i=1}^{M}$ for $\mathcal{H}_N$ is called scalable or scalable frame if there exist scalars $\{c_i\}_{i=1}^{M}$ such that $\{c_i\varphi_i\}_{i=1}^{M}$ is a tight frame for $\mathcal{H}_N$.

In the definition of scalable frame, the scalars c_i usually satisfy $|c_i|^2 \geq 0$.

Example 5 The frame is Example 3 is a scalable frame with $c_1 = 1 = c_3, c_2 = 2$.

Thus, if a frame is scalable, there exist scalars such that the scaled frame is a Parseval frame. Therefore using Theorem 1, one can conclude that a frame $\Phi = \{\varphi_i\}_{i=1}^{M}$ is scalable if and only if there exists $c_i \geq 0$ such that

$$I_N = \sum_{i=1}^{M} c_i \varphi_i \varphi_i^*.$$

Kutyniok et al. introduced the notion of a scalable frame in 2013. some characterizations of scalable frames, both of functional analytic and geometric type were derived in the infinite and finite dimensional settings by them. Considering the case of complex, it was proved that the set of all possible sequences of scalars is the convex hull of minimal scalars. The numerical aspects of scalable frames are discussed recently. Using numerical algorithms, different scalings are computed with different purposes in applications. Recently, some work has been done to characterize frames which are not scalable. The notion of frame scaling has been extended to matrices also with Laurent polynomials having applications to construct tight wavelet filter banks.

Once it is known that there are some frames which are not scalable, then one look for the optimal condition i.e. how close the constants A and B can be and what are choices of the scaling scalars to attain this closeness. In other words, if a frame is not scalable, with which scalars one can make the frame as tight as possible. Naturally it means that $\{c_i\varphi_i\}_{i=1}^M$ is the optimal or best conditioned in the sense that the ratio of upper frame bound and lower frame bound is the closest to 1. Some work has been done in this direction recently by Cassaza and Chen.

3. Frame Scaling in Finite Dimensional Hilbert Spaces

In this section, the notion of frame scaling is considered in finite dimensional Hilbert spaces. Tight frames in finite dimensional Hilbert spaces possess some beautiful geometrical properties and play an important role in applications.

We prove the following result concerning frame scaling in finite dimensional Hilbert spaces.

Theorem 2. Let $\Phi = \{\varphi_i\}_{i=1}^N$ be a frame for $\mathcal{H}_N$. Let $\{e_i\}_{i=1}^N$ be the orthonormal basis of $\mathcal{H}_N$. If for each i, $[\varphi_n]_{n=1}^i = [e_n]_{n=1}^i$, then the frame $\Phi = \{\varphi_i\}_{i=1}^N$ is scalable.

Proof. Let $\Phi = \{\varphi_i\}_{i=1}^N$ be a frame for $\mathcal{H}_N$. Let $\{e_i\}_{i=1}^N$ be the orthonormal basis of $\mathcal{H}_N$. Without any loss of generality, we can assume that $\varphi_i \neq 0$, for any i. We are given that for each i, $[\varphi_n]_{n=1}^i = [e_n]_{n=1}^i$.
This gives

$$[\varphi_1] = [e_1], [\varphi_1, \varphi_2] = [e_1, e_2], \cdots, [\varphi_1, \varphi_2, \cdots, \varphi_N] = [e_1, e_2, \cdots, e_N].$$

Then, for each i, the φ_i can be expressed in terms of e_i.
Therefore, for each i, there exists a non-zero scalar a_i such that $\varphi_i = a_i e_i$.
Choose, $c_i = \frac{1}{a_i}$. Then $\{c_i\varphi_i\}_{i=1}^N$ is a tight frame for $\mathcal{H}_N$.
Hence $\Phi = \{\varphi_i\}_{i=1}^N$ is a scalable frame.

Example 6. Let $\mathcal{H} = \mathbb{R}^3(\mathbb{R})$, the 3-dimensional space.
Let $\Phi = \{\varphi_1 = (\frac{1}{2}, 0, 0), \varphi_2 = (0, \frac{1}{2}, 0), \varphi_3 = (0, 0, 1)\}$. Then

$$\frac{1}{2}\|x\|^2 \leq \sum_{i=1}^{3} |\langle x, \varphi_i\rangle|^2 \leq \|x\|^2.$$

So, taking $A = \frac{1}{2}, B = 1$, Φ is an exact frame for $\mathcal{H}$. But it is not tight and not unit norm frame. Choosing $c_1 = c_2 = 2, c_3 = 1$, the frame can be made tight and hence it is a scalable frame for $\mathcal{H} = \mathbb{R}^3(\mathbb{R})$.
Example 7. Let $\mathcal{H} = \mathbb{R}^3(\mathbb{R})$, the 3-dimensional space.
Let $\Phi = \{\varphi_1 = (\frac{1}{2}, 0, 0), \varphi_2 = (\frac{1}{2}, 0, 0), \varphi_3 = (0, 1, 0), \varphi_4 = (0, 0, 1)\}$. Then

$$\|x\|^2 \leq \sum_{i=1}^{4} |\langle x, \varphi_i\rangle|^2 \leq \|x\|^2.$$

So, taking $A = B = 1$, Φ is a tight frame for $\mathcal{H}$. But it is not exact and not unit norm frame. Choosing $c_1 = c_2 = 2, c_3 = 1, c_4 = 0$ or $c_1 = c_2 = 2, c_3 = \frac{1}{2} = c_4$, the frame can be made tight and hence it is a scalable frame for $\mathcal{H} = \mathbb{R}^3(\mathbb{R})$.

Similarly, one can construct other types of frames in infinite dimensional separable Hilbert spaces and finite dimensional Hilbert spaces. One can note that these are orthogonal basis in a finite dimensional Hilbert space (Example 6) and the frame is inexact if it has more number of elements in the collection than the dimension of the space (Example 7) but there is a variation in orthogonal set up here in the example.

In fact, with similar approach, one can prove a slightly general result. We state it here and the proof follows in similar fashion.

Theorem 3. Let $\Phi = \{\varphi_i\}_{i=1}^M$ be a frame for $\mathcal{H}_N$. Let $\{e_i\}_{i=1}^N$ be the orthonormal basis of $\mathcal{H}_N$. If for each i, φ_i is can be expressed in term of e_j, for some j, then the frame $\Phi = \{\varphi_i\}_{i=1}^M$ is scalable.

Using Theorem 2 and Theorem 3, one can conclude that the frames in Example 3 and Example 4 are scalable frames.

Given a frame, we usually talk of its dual. The dual of a frame possesses all the properties of the frame with slight modifications. What can we say about the scalability of the dual of a scalable frame? Regarding the scalability of duals, the following result is obtained.

Theorem 4. Let $\Phi = \{\varphi_i\}_{i=1}^M$ be a frame for $\mathcal{H}_N$. Let $\Phi^* = \{\varphi_i^*\}_{i=1}^M$ be its dual. Then the frame $\Phi = \{\varphi_i\}_{i=1}^M$ is scalable if and only if the frame $\Phi^* = \{\varphi_i^*\}_{i=1}^M$ is scalable.

Proof. We need to prove that $\Phi^* = \{\varphi_i^*\}_{i=1}^M$ is scalable if $\Phi = \{\varphi_i\}_{i=1}^M$ is scalable. The dual of the dual is the frame itself, the converse will follow obviously.

Let $\Phi = \{\varphi\}_{i=1}^M$ be scalable. Then there exists $\{c_i\}_{i=1}^M$ such that $\{c_i\varphi_i\}_{i=1}^M$ is a tight frame for $\mathcal{H}_N$.

If any of $c_i = 0$, then we can drop the corresponding element φ_i from the collection $\Phi = \{\varphi\}_{i=1}^M$ and the rest of the collection is again a tight frame for $\mathcal{H}_N$.

Therefore, without any loss, we may assume that non of c_i is zero.

Since $\Phi = \{\varphi\}_{i=1}^M$ is a frame for $\mathcal{H}_N$, then for any $f \in \mathcal{H}_N$, we have

$$S(f) = \sum_{i=1}^{M} \langle f, \varphi_i^* \rangle \varphi_i,$$

where S is the frame operator of the frame φ_i.

It gives

$$f = \sum_{i=1}^{M} \left\langle f, S^{-1}\varphi_i^* \right\rangle \varphi_i.$$

As we have assumed that $\Phi = \{\varphi\}_{i=1}^M$ is scalable, therefore for any $f \in \mathcal{H}_N$, we have

$$f=\sum_{i=1}^{M}\langle f,\varphi_i^{**}\rangle c_i\varphi_i,$$

where φ_i^{**} is the dual frame of the frame $c_i\varphi_i$.

This implies

$$f=\sum_{i=1}^{M}\left\langle f,\frac{1}{c_i}\varphi_i^{**}\right\rangle \varphi_i.$$

This implies $\left\{\frac{1}{c_i}\varphi\right\}_{i=1}^{M}$ is a tight frame for $\mathcal{H}_N$ and hence $\{\varphi_i^*\}$ is a scalable frame.

This completes the proof.

Example 8. Let $\mathcal{H}=\mathbb{R}^3(\mathbb{R})$, the 3-dimensional space.
Let $\Phi=\big\{\varphi_1=(1,0,0),\varphi_2=(0,\frac{1}{2},0),\varphi_3=(0,0,1)\big\}$. Then

$$\frac{1}{2}\|x\|^2\leq\sum_{i=1}^{3}|\langle x,\varphi_i\rangle|^2\leq\|x\|^2.$$

So, taking $A=\frac{1}{2},B=1$, Φ is an exact frame for $\mathcal{H}$. But it is not tight and not unit norm frame. Its dual frame is given by $\Phi^*=\{\varphi_1^*=(1,0,0),\varphi_2^*=(0,2,0),\varphi_3^*=(0,0,1)\}$. Frame and its dual frame are scalabale.

Example 9. Let $\mathcal{H}=\mathbb{R}^3(\mathbb{R})$, the 3-dimensional space.
Let $\Phi=\big\{\varphi_1=(\frac{1}{2},0,0),\varphi_2=(\frac{1}{2},0,0),\varphi_3=(0,1,0),\varphi_4=(0,0,1)\big\}$. Then

$$\|x\|^2\leq\sum_{i=1}^{4}|\langle x,\varphi_i\rangle|^2\leq\|x\|^2.$$

So, taking $A=B=1$, Φ is a tight frame for $\mathcal{H}$. But it is not exact and not unit norm frame. Its dual is given by $\Phi^*=\big\{\varphi_1^*=(2,0,0),\varphi_2^*=(2,0,0),\varphi_3^*=(0,\frac{1}{2},0),\varphi_4^*=(0,0,\frac{1}{2})\big\}$. Both the frames are scalable.

Similarly, one can construct other types of frames in infinite dimensional separable Hilbert spaces and finite dimensional Hilbert spaces.

In the next result, we show that union of two scalable frames is a scalable frame.

Theorem 5. Let $\Phi_1=\big\{\varphi_i^1\big\}_{i=1}^{M_1}$ and $\Phi_2=\big\{\varphi_i^2\big\}_{i=1}^{M_2}$ be scalable frames for $\mathcal{H}_N$. Then $\Phi_1\cup\Phi_2$ is a scalable frame for $\mathcal{H}_N$.

Proof. Since $\Phi_1=\big\{\varphi_i^1\big\}_{i=1}^{M_1}$ and $\Phi_2=\big\{\varphi_i^2\big\}_{i=1}^{M_2}$ are scalable frames for $\mathcal{H}_N$, then there exist scalars $\{c_i\}_{i=1}^{M_1}$ and $\{d_i\}_{i=1}^{M_2}$ such that $\big\{c_i\varphi_i^1\big\}_{i=1}^{M_1}$ and $\big\{d_i\varphi_i^2\big\}_{i=1}^{M_2}$ are tight frames for $\mathcal{H}_N$.

This gives

$$f=\sum_{i=1}^{M_1}\big\langle f,S_1^{-1}\varphi_i^{1*}\big\rangle c_i\varphi_i$$

and

$$f = \sum_{i=1}^{M_2} \left\langle f, S_2^{-1}\varphi_i^{2*}\right\rangle d_i\varphi_i,$$

where S_1, S_2 are respective frame operators and $\left\{\varphi_i^{1*}\right\}_{i=1}^{M_1}$, $\left\{\varphi_i^{2*}\right\}_{i=1}^{M_2}$ are respective dual frames of $\Phi_1 = \left\{\varphi_i^1\right\}_{i=1}^{M_1}$, $\Phi_2 = \left\{\varphi_i^2\right\}_{i=1}^{M_2}$.

Consider $\Phi = \Phi_1 \cup \Phi_2 = \left\{\varphi_i^{1*}\right\}_{i=1}^{M_1} \cup \left\{\varphi_i^{2*}\right\}_{i=1}^{M_2} = \{\varphi_i\}_{i=1}^{M_1+M_2}$.

Choose scalars $\{e_i\}_{i=1}^{M_1+M_2}$ such that

$$e_i = \frac{c_i}{2},$$

if $1 \leq i \leq M_1$ and

$$e_i = \frac{d_i}{2},$$

if $M_1 + 1 \leq i \leq M_2$.

Then

$$\sum_{i=1}^{M_1} \left\langle f, S_1^{-1}\varphi_i^{1*}\right\rangle \frac{c_i}{2}\varphi_i^1 + \sum_{i=1}^{M_2} \left\langle f, S_2^{-1}\varphi_i^{2*}\right\rangle \frac{d_i}{2}\varphi_i^2 = \frac{f}{2} + \frac{f}{2} = f.$$

But

$$\sum_{i=1}^{M_1} \left\langle f, S_1^{-1}\varphi_i^{1*}\right\rangle \frac{c_i}{2}\varphi_i^1 = \sum_{i=1}^{M_1} \left\langle f, S_1^{-1}\varphi_i^{1*}\right\rangle \frac{e_i}{2}\varphi_i^1$$

and

$$\sum_{i=1}^{M_1} \left\langle f, S_1^{-1}\varphi_i^{1*}\right\rangle \frac{c_i}{2}\varphi_i^1 = \sum_{i=M_1+1}^{M_1+M_2} \left\langle f, S_1^{-1}\varphi_i^{2*}\right\rangle \frac{e_i}{2}\varphi_i^2.$$

This gives

$$f = \sum_{i=1}^{M_1+M_2} \left\langle f, S_1^{-1}\varphi_i^{*}\right\rangle e_i\varphi_i.$$

Hence $\Phi = \Phi_1 \cup \Phi_2 = \left\{\varphi_i^{1*}\right\}_{i=1}^{M_1} \cup \left\{\varphi_i^{2*}\right\}_{i=1}^{M_2} = \{\varphi_i\}_{i=1}^{M_1+M_2}$ is a scalable frame for $\mathcal{H}_N$.

Example 10. Let $\mathcal{H} = \mathbb{R}^3(\mathbb{R})$, the 3-dimensional space.
Let $\Phi = \left\{\varphi_1 = (1,0,0), \varphi_2 = (0,\frac{1}{2},0), \varphi_3 = (0,0,1)\right\}$. Then

$$\frac{1}{2}\|x\|^2 \leq \sum_{i=1}^{3} |\langle x, \varphi_i\rangle|^2 \leq \|x\|^2.$$

So, taking $A = \frac{1}{2}, B = 1$, Φ is an exact frame for $\mathcal{H}$. But it is not tight and not unit norm frame.

Let us consider another frame. Let $\mathcal{H} = \mathbb{R}^3(\mathbb{R})$, the 3-dimensional space.
Let $\Phi = \left\{\varphi_1 = (\frac{1}{2}, 0, 0), \varphi_2 = (\frac{1}{2}, 0, 0), \varphi_3 = (0, 1, 0), \varphi_4 = (0, 0, 1)\right\}$. Then

$$\|x\|^2 \leq \sum_{i=1}^{4} |\langle x, \varphi_i\rangle|^2 \leq \|x\|^2.$$

So, taking $A = B = 1$, Φ is a tight frame for $\mathcal{H}$. But it is not exact and not unit norm frame. Both the frames are scalable and if we consider the union, then we may check out easily that the union is also a scalable frame.

Similarly, one can construct other types of frames in infinite dimensional separable Hilbert spaces and finite dimensional Hilbert spaces.

We can generalize Theorem 5 to any finite index. The proof can given on the similar lines. We state the result without its proof.

Theorem 6. Let for any natural number n, $\Phi_1 = \left\{\varphi_i^1\right\}_{i=1}^{M_1}$, $\Phi_2 = \left\{\varphi_i^2\right\}_{i=1}^{M_2}$, $\Phi_3 = \left\{\varphi_i^3\right\}_{i=1}^{M_3}$ ... and $\Phi_n = \{\varphi_i^n\}_{i=1}^{M_n}$ be scalable frames for $\mathcal{H}_N$. Then $\Phi_1 \cup \Phi_2 \cup \Phi_3 \cup ... \cup \Phi_n$ is a scalable frame for $\mathcal{H}_N$.

Scalable frames or frame scaling is being studied in context of frame potential recently. The notion of frame potential is widely used in physical sciences. One may explore to check if a sub-collection of a scalable frame is scalable frame sequence or not. There are many other directions in which lot of research work is being carried out by various researchers.

Conclusion

In this short note, we have discussed and studied frames in finite dimensional Hilbert. Frame scaling of frames in finite dimensional Hilbert spaces is discussed and studied and few results regarding frame scaling in finite dimensional Hilbert spaces are obtained. Dual frames of frames are considered and it is proved that a frame is scalable if and only if its dual frame is scalable. Union of two frames is considered and it is proved that if two frames are scalable, then the union is also a scalable frame. Illustrative examples are provided in the note. Some other directions related to scalable frames are mentioned in the note, that may be explored for further investigation of the topic.

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