EXPANSION FORMULAE INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTION

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Abstract

The aim of the paper is to establish a general expansion formula for the Aleph – function of one variable. It is significiant to observe that a large number of finite and infinite series for this function can be easily summed up by using the summation theorem for ordinary hypergeometric series in the main result. Also, by appropriately specializing the parameters of the aleph function, one can obtain expansion formulas for simpler special functions of one variables.

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1. Introduction :

In this chapter, we have established some finite and infinite series expansion formulae involving $\aleph$-function. the results obtained here are interesting and of general character and hence encompass several cases of interest.McRobert [1958,1959,1959a],Srivastava H.M.[1985], Srivastava H.M. and Panda R.[1975], Ayant F.Y.[2016] ,Agrawal M.K and Jain S.S.L.[1997] Anandani,P.[1969] ,Sharma ,C.K. and Jain , N.C. [1972] ,Sharma , C.K. and Chouksey,S.K. [1985], and other established the finite and infinite expansion formulae for the E and H function.

Here, we have obtained sum of number of finite and infinite series involving multivariable ℵ-function by expressing the ℵ-function as Mellin - Barnes type integral (), interchanging the order of integration and summation and then evaluating the summation inside the integral with the help of known results. These known results are basically the known theorems like Saalshutz, Dixon, Watson etc. and some identities like Gould identity, Lagrange's expansion, formulae etc. Remaining results have been derived with the help of known results due to Bhatt [1965].

Recently Vyas and Rathie [1999] established an expansion formula involving Gagebauer Polynomials and mul ℵ-function.

2. EXPANSION FORMULAE :

The following expansion formulae can be established satisfying the following condition.

(1) $\alpha, \beta, s \geq 0$

(2) $\eta \geq o, z \geq 0$ and

(3) $|\arg \mathrm{z}| < 1/2\ \mathrm{V}\pi, V > 0$, the definition of $\aleph-$ function.

$$\sum_{k=0}^{Z} \frac{(-z)_k \left(\alpha - z - \frac{1}{2}\right)_k}{(k)! \left(\frac{3}{2} - \alpha\right)_k} \aleph^{m+1,n+1}_{p_i+2,q_i+2,\tau_i;r} \left[Z \left| \begin{array}{c} (\alpha - k, h), \ldots\ldots, (\alpha + k - z, h) \\ (k, h), \ldots., \ldots, (-k - z, h) \end{array} \right. \right]$$
$$= \frac{2^{2z}(1-\alpha)_z}{(2-2\alpha)_z} \aleph^{m+3,n+3}_{p_i+6,q_i+6,\tau_i;r}$$
$$\left[Z \begin{array}{c} (\alpha, h), \left(\alpha - \frac{z}{2}, h\right), \left(\alpha - \frac{z+1}{2}, h\right), \ldots\ldots\ldots, (\alpha - z, h), \left(\frac{1}{2}, h\right), (z, h) \\ (0, h), \left(\frac{z}{2}, h\right), \ldots, \ldots, (-k, h) \left(\alpha - \frac{1}{2}, h\right), (\alpha - z, h) \end{array} \right], \quad (2.1)$$

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$$\sum_{k=0}^{Z}\frac{(-z)_k\left(\alpha-z-\frac{1}{2}\right)_k}{(k)!\left(\frac{3}{2}-\alpha\right)_k}\aleph^{m,n+2}_{p_i+2,q_i+2,\tau_i;r}\left[Z\left|\begin{array}{c}(\alpha-k-z,h),(\alpha-k,h),\ldots\ldots.,\ldots\ldots..\\ \ldots\ldots,\ldots\ldots,(k,h),(1+z-k,h)\end{array}\right.\right]$$
$$=\frac{2^{2z}(1-\alpha)_z}{(2-2\alpha)_z}\aleph^{m+2,n+3}_{p_i+5,q_i+5,\tau_i;r}$$
$$\left[z\left|\begin{array}{c}(\alpha,h),\left(\alpha-\frac{z}{2},h\right),\left(\alpha-\frac{z+1}{2},h\right),\ldots,\ldots\ldots,\left(\frac{1}{2},h\right),(1+z,h)\\ ,\left(\frac{z+1}{2},h\right),\left(1+\frac{z}{2},h\right)\ldots,\ldots,(0,h)\left(\alpha-\frac{1}{2},h\right),(1+z,h)\end{array}\right.\right], \tag{2.2}$$

Proof :

To prove (2.1), we first replace the $\aleph$-function in left -hand - side .by its Mellin -Barnes type contour integral

$$\sum_{k=0}^{Z}\frac{(-z)_k\left(\alpha-z-\frac{1}{2}\right)_k}{(k)!\left(\frac{3}{2}-\alpha\right)_k}\frac{1}{2\pi i}\int_L\Omega^{m,n}_{p_i,q_i,\tau_i;r}(\xi)\frac{\Gamma(k-h\xi),\Gamma(1-\alpha+k+h\xi)}{\Gamma(1+z+k+h\xi),\lceil(\alpha+k-z,h\xi)}z^{-\xi}d\xi$$

Interchanging the order of summation and integration,

$$\frac{1}{2\pi i}\int_L\Omega^{m,n}_{p_i,q_i,\tau_i;r}(\xi)z^{-\xi}d\xi\left\{\sum_{k=o}^{z}\frac{(-z)_k\left(\alpha-z-\frac{1}{2}\right)k\Gamma(k-h\xi)\Gamma(1-\alpha+k+h\xi)}{(k)!\left(\frac{3}{2}-\alpha\right)_k\Gamma(1+z+k+h\xi)\Gamma(\alpha+k-z-h\xi)}\right\}$$

Using $(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$, here, we have

$$\frac{1}{2\pi i}\int_L\Omega^{m,n}_{p_i,q_i,\tau_i;r}(\xi)\frac{\Gamma(0-h\xi)\Gamma(1-\alpha+h\xi)}{\Gamma(1+z+h\xi)\Gamma(\alpha-z-h\xi)}z^{-\xi}d\xi$$

$$\left\{\sum_{k=0}^{z}\frac{(-z)_k\left(\alpha-z-\frac{1}{2}\right)_k\Gamma(k-h\xi)\Gamma(1-\alpha+k+h\xi)}{(k)!\left(\frac{3}{2}-\alpha\right)_k\Gamma(1+z+h\xi)_k\Gamma(\alpha-z-h\xi)_k}\right\}$$
$$=\frac{1}{2\pi i}\int_L\Omega^{m,n}_{p_i,q_i,\tau_i;r}(\xi)\frac{\Gamma(0-h\xi)\Gamma(1-\alpha+h\xi)}{\Gamma(1+z+h\xi)\Gamma(\alpha-z-h\xi)}$$
$$4^{F_3}\left[\begin{array}{c}-z,1-\alpha+h\xi,0-h\xi,\alpha-z-\frac{1}{2}\\ \alpha-z-h\xi,1+z+h\xi,\frac{3}{2}-\alpha\end{array}\left|1\right.\right]z^{-\xi}d\xi.$$
$$Using4^{F_3}\left[\begin{array}{c}-n,a,b,1/2-a-b-n\\ 1-a-n,1-b-n,1/2+a+b\end{array}\left|1\right.\right]=\frac{(2a)_n(2b)_n(a+b)_n}{(a)_n(b)_n(2a+2b)_n}$$

here , we get

$$\frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(\xi)$$
$$\frac{\Gamma(1-\alpha+h\xi)\Gamma(0-h\xi)(2-2\alpha+2h\xi)_z(-2h\xi)_z(1-\alpha)_z}{\Gamma(1+z+h\xi)\Gamma(\alpha-z-h\xi)(2-2\alpha)_z(1-\alpha+h\xi)_z(-h\xi)_z}Z^{-\xi}d\xi$$

Again using $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ and Legendre Duplication Formula here,we get

$$\frac{2^{2z}(1-\alpha)_z}{(2-2\alpha)_z}\frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(\xi)$$
$$\frac{\Gamma(0-h\xi)\Gamma(1-\alpha+h\xi)\Gamma(1-\alpha+\frac{z}{2}+h\xi)\Gamma(1-\alpha+\frac{z+1}{2}+h\xi)}{\Gamma(1+z+h\xi)\Gamma(1-\alpha+h\xi)\Gamma(1-\alpha+\frac{1}{2}+h\xi)}$$
$$\times\frac{\Gamma\left(\frac{z}{2}-h\xi\right)\Gamma\left(\frac{z+1}{2}-h\xi\right)\Gamma(1-\alpha+h\xi)\Gamma(0-h\xi)}{\Gamma(0-h\xi)\Gamma\left(\frac{1}{2}-h\xi\right)\Gamma(1-\alpha+z+h\xi)\Gamma(z-h\xi)}z^{-\xi}d\xi$$

$$\frac{2^{2z}(1-\alpha)_z}{(2-2\alpha)_z}\frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(\xi)\frac{\Gamma(0-h\xi)\ulcorner(\frac{z}{2}-h\xi)\left\lceil\left(\frac{z+1}{2}-h\xi\right)\right.}{\Gamma(1+z+h\xi)\Gamma\left(1-\alpha+\frac{1}{2}+h\xi\right)}$$
$$\times\frac{\Gamma\left(1-\alpha+\frac{z}{2}+h\xi\right)\ulcorner\left(1-\alpha+\frac{z+1}{2}+h\xi\right)\ulcorner(1-\alpha+h\xi)}{\Gamma(1-\alpha+z+h\xi)\Gamma(\alpha-z-h\xi)\Gamma(z-h\xi)}z^{-\xi}d\xi$$

$$=\frac{2^{2z}(1-\alpha)_z}{(2-2\alpha)_z}\aleph^{m+3,n+3}_{p_i+6,q_i+6,\tau_i;r}\left[\begin{array}{c}(\alpha,h),\left(\alpha-\frac{z}{2},h\right),\left(\alpha-\frac{z+1}{2},h\right),\ldots,\ldots,(\alpha-z,h),\left(\frac{1}{2},h\right),(z,h)\\(0,h),\left(\frac{z}{2},h\right),\ldots,\ldots,(-k,h)\left(\alpha-\frac{1}{2},h\right),(\alpha-z,h)\end{array}\right],$$

Which is the required result.

To prove (2.2), expressing $\aleph$-function in left -hand - side as contour integral

$$\sum_{k=0}^{z}\frac{(-z)_k\left(\alpha-z-\frac{1}{2}\right)k}{(k)!\left(\frac{3}{2}-\alpha\right)_k}\frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(\xi)\frac{\Gamma(1-\alpha-k+z+h\xi),\Gamma(1-\alpha+k+h\xi)}{\Gamma(1-k+h\xi),\Gamma(1-z+k,h\xi)}z^{-\xi}d\xi$$

Interchanging the order of summation and integration,

$$\frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i r}(\xi)z^{-\xi}d\xi\left\{\sum_{k=o}^{z}\frac{(-z)_k\left(\alpha-z-\frac{1}{2}\right)k\Gamma(1-\alpha+z+h\xi-k)\Gamma(1-\alpha+k+h\xi)}{(k)!\left(\frac{2}{2}-\alpha\right)_k\Gamma(1-h\xi-k)\Gamma(1-z+h\xi+k)}\right\}.$$

Using $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ and $(\alpha)_n = \frac{(-1)^n\Gamma(1-\alpha)}{\Gamma(1-\alpha-n)}$,
here, we have

$$\frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(\xi)\frac{\Gamma(1-\alpha+z+h\xi)\Gamma(1-\alpha+h\xi)}{\Gamma(1+h\xi)\Gamma(1-z-h\xi)}z^{-\xi}d\xi$$

$$\left.\sum_{k=o}^{z}\frac{(-z)_k\left(\alpha-z-\frac{1}{2}\right)_k\Gamma(1-\alpha-h\xi)_k\Gamma(-h\xi)_k}{(k)!\left(\frac{3}{2}-\alpha\right)_k\Gamma(\alpha-z-h\xi)_k\Gamma(-z+h\xi)_k}\right\}$$

$$=\frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(\xi)\frac{\Gamma(1-\alpha+z+h\xi)\Gamma(1-\alpha+h\xi)}{\Gamma(1+h\xi)\Gamma(1-z+h\xi)}$$

$$4^{F_3}\left[\begin{array}{c} -z,1-\alpha+h\xi,0-h\xi,\alpha-z-\frac{1}{2} \\ \alpha-z-h\xi,-z+h\xi,\frac{3}{2}-\alpha \end{array}\middle|\,1\right]Z^{-\xi}d\xi.$$

$$U\,\text{sing}\;4^{F_3}\left[\begin{array}{c} -n,a,b,1/2-a-b-n \\ 1-a-n,1-b-n,a+b-\frac{1}{2} \end{array}\middle|\,1\right]=\frac{(2a)_n(2b)_n(a+b)_n}{(a)_n(b)_n(2a+2b-1)_n}$$

here, we get

$$\frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(\xi)\frac{\Gamma(1-\alpha+z+h\xi)\Gamma(1-\alpha+h\xi)(2-2\alpha+2h\xi)_z(1-2h\xi)_z(1-\alpha)_z}{\Gamma(1+h\xi)\Gamma(-z+h\xi)(1-\alpha+h\xi)_z(1-h\xi)_z(2-2\alpha)_z}Z^{-\xi}d\xi$$

Again using $(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ and Legendre Duplication Formula
here, we get

$$\frac{2^{2z}(1-\alpha)_z}{(2-2\alpha)_z}\frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(\xi)$$

$$\frac{\Gamma(1-\alpha+z+h\xi)\Gamma(1-\alpha+h\xi)\Gamma(1-\alpha+\frac{z}{2}+h\xi)\Gamma(1-\alpha+\frac{z+1}{2}+h\xi)\Gamma(1-\alpha+\frac{z+2}{2}+h\xi)}{\Gamma(1+z+h\xi)\Gamma(1-\alpha+h\xi)\Gamma\left(1-\alpha+\frac{1}{2}+h\xi\right)}$$

$$=\frac{2^{2z}(1-\alpha)_z}{(2-2\alpha)_z}\frac{1}{2\pi i}\Omega^{m,n}_{p_i,q_i,\tau_i;r}(\xi)$$

$$\frac{\Gamma\left(\frac{z+1}{2}-h\xi\right)\left(1+\frac{z}{2}-h\xi\right)\lceil(1-\alpha+h\xi)}{\Gamma(1-0+h\xi)\lceil\{1-(1+z)+h\xi)\lceil\left\{1-\left(\alpha-\frac{1}{2}\right)+h\xi\right\}}$$

$$\times\frac{\Gamma\left\{1-\left(\alpha-\frac{z}{2}\right)+h\xi\right\}\lceil\left\{1-\left(\alpha-\frac{z+1}{2}\right)+h\xi\right\}}{\Gamma\left(\frac{1}{2}-h\xi\right)\lceil(1+z-h\xi)}z^{-\xi}d\xi$$

$$=\frac{2^{2z}(1-\alpha)_z}{(2-2\alpha)_z}\aleph^{m+2,n+3}_{p_i+5,q_i+5,\tau_i;r}\left[z\,\middle|\,\begin{array}{c}(\alpha,h),\left(\alpha-\frac{z}{2},h\right),\left(\alpha-\frac{z+1}{2},h\right),\ldots,\ldots\ldots,\left(\frac{1}{2},h\right),(1+z,h)\\ ,\left(\frac{z+1}{2},h\right),\left(1+\frac{z}{2},h\right)\ldots,\ldots,(0,h)\left(\alpha-\frac{1}{2},h\right),(1+z,h)\end{array}\right],$$

Which is the required result.

3. SPECIAL CASES OF (2.1) and (2.2)

(1) On taking $r = 1$, the result (2.1) reduces to

$$\sum_{k=0}^{Z} \frac{(-z)_k \left(\alpha - z - \frac{1}{2}\right)_k}{(k)! \left(\frac{3}{2} - \alpha\right)_k} H^{m+1,n+1}_{p+2,q+2} \left[Z \middle| \begin{array}{c} (\alpha - k, h), \ldots\ldots, (\alpha + k - z, h) \\ (k, h), \ldots, \ldots, (-k - z, h) \end{array} \right]$$
$$= \frac{2^{2z}(1-\alpha)_z}{(2-2\alpha)_z} H^{m+3,n+3}_{p+6,+6} \left[z \middle| \begin{array}{c} (\alpha, h), \left(\alpha - \frac{z}{2}, h\right), \left(\alpha - \frac{z+1}{2}, h\right), \ldots, \ldots, (\alpha - z, h), \left(\frac{1}{2}, h\right), (z, h) \\ (0, h), \left(\frac{z}{2}, h\right), \ldots, \ldots, (-k, h) \left(\alpha - \frac{1}{2}, h\right), (\alpha - z, h) \end{array} \right], \tag{3.1}$$

Further , on taking $\alpha_j = \beta_j = 1$ result reduces to that of G - function.

$$\sum_{k=0}^{Z} \frac{(-z)_k \left(\alpha - z - \frac{1}{2}\right)_k}{(k)! \left(\frac{3}{2} - \alpha\right)_k} G^{m+1,n+1}_{p+2,q+2} \left[z \middle| \begin{array}{c} (\alpha - k, h), \ldots\ldots, (\alpha + k - z, h) \\ (k, h), \ldots, \ldots, (-k - z, h) \end{array} \right]$$
$$= \frac{2^{2z}(1-\alpha)_z}{(2-2\alpha)_z} G^{m+3,n+3}_{p+6,+6} \left[z \middle| \begin{array}{c} (\alpha, h), \left(\alpha - \frac{z}{2}, h\right), \left(\alpha - \frac{z+1}{2}, h\right), \ldots, \ldots, (\alpha - z, h), \left(\frac{1}{2}, h\right), (z, h) \\ (0, h), \left(\frac{z}{2}, h\right), \ldots, \ldots, (-k, h) \left(\alpha - \frac{1}{2}, h\right), (\alpha - z, h) \end{array} \right], \tag{3.2}$$

(2) On taking $r = 1$, the result (2.2) reduces to

$$\sum_{k=0}^{Z} \frac{(-z)_k \left(\alpha - z - \frac{1}{2}\right)_k}{(k)! \left(\frac{3}{2} - \alpha\right)_k} H^{m,n+2}_{p_i+2,q_i+2,\tau_i;r} \left[Z \middle| \begin{array}{c} (\alpha - k - z, h), (\alpha - k, h), \ldots\ldots, \ldots\ldots \\ \ldots\ldots, \ldots, (k, h), (1 + z - k, h) \end{array} \right]$$
$$= \frac{2^{2z}(1-\alpha)_z}{(2-2\alpha)_z} H^{m+2,n+3}_{p_i+5,q_i+5,\tau_i;r} \left[z \middle| \begin{array}{c} (\alpha, h), \left(\alpha - \frac{z}{2}, h\right), \left(\alpha - \frac{z+1}{2}, h\right), \ldots, \ldots, \left(\frac{1}{2}, h\right), (1 + z, h) \\ , \left(\frac{z+1}{2}, h\right), \left(1 + \frac{z}{2}, h\right), \ldots, \ldots, (0, h) \left(\alpha - \frac{1}{2}, h\right), (1 + z, h) \end{array} \right], \tag{3.3}$$

Further , on taking $\alpha_j = \beta_j = 1$ result reduces to that of G - function.

$$\sum_{k=0}^{Z} \frac{(-z)_k \left(\alpha - z - \frac{1}{2}\right)_k}{(k)! \left(\frac{3}{2} - \alpha\right)_k} G^{m,n+2}_{p_i+2,q_i+2,\tau_i;r} \left[z \middle| \begin{array}{c} (\alpha - k - z, h), (\alpha - k, h), \ldots, \ldots \\ \ldots, \ldots, (k, h), (1 + z - k, h) \end{array} \right]$$
$$= \frac{2^{2z}(1-\alpha)_z}{(2-2\alpha)_z} G^{m+2,n+3}_{p_i+5,q_i+5,\tau_i;r} \left[z \middle| \begin{array}{c} (\alpha, h), \left(\alpha - \frac{z}{2}, h\right), \left(\alpha - \frac{z+1}{2}, h\right), \ldots, \ldots, \left(\frac{1}{2}, h\right), (1 + z, h) \\ , \left(\frac{z+1}{2}, h\right), \left(1 + \frac{z}{2}, h\right), \ldots, \ldots, (0, h) \left(\alpha - \frac{1}{2}, h\right), (1 + z, h) \end{array} \right], \tag{3.4}$$

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