Exp-gamma Distribution as a New Life Time Model and its Applications

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Abstract

In this paper, a new distribution namely exp-gamma distribution is proposed. The different mathematical and statistical properties of the proposed distribution are derived and discussed. The survival function, hazard rate function and mean residual life function of the proposed distribution is discussed. The expression for Bonferroni and Lorenz curves of the proposed distribution are obtained. The parameters of the proposed distribution are estimated by using method of maximum likelihood estimation. Performance of the proposed model is tested using real life data sets.

Keywords: Exp-gamma Distribution; Moments; Moment Generating Function; Reliability Analysis; Parameter Estimation.

1. Introduction

A mixture distribution is convex combination of few or more probability density functions. The distributions combined to form a mixture distribution are called as mixture components and the probability associated with each component is called the mixture weight. Mixture distributions are used to analyze complex data with greater flexibility when two or more probability density functions are combined. Mixture distribution is frequently used in statistical analysis including modelling, classification and survival analysis. Newcomb (1886) used the concept of finite mixture distribution while modelling outlier. Kao (1959) used a finite mixture of Weibull distribution for life testing of electric tubes. Several researchers have studied mixture distribution. For more details reader can refer Nakhi and Kalla (2004), Everitt and Hand (1981), Safiq et.al. (2022), Daghestani et.al. (2021).
Shanker and Shukla (2017) have introduced Ishita distribution and Shanker et.al. (2018) have introduced Akash distribution. Both the distributions are mixture distributions of exponential and gamma distribution with mixing proportion $\frac{\theta^3}{\theta^3+3}$ and $\frac{\theta^2}{\theta^3+2}$ respectively.

In this paper a new life time distribution is proposed named as, Exp-gamma distribution. The proposed distribution is defined as follows,

A non-negative continuous random variable X is said to follow Exp-gamma distribution with parameter θ , if its probability density function is of the form,

$$f(x;\theta) = \frac{\theta}{\theta^3+2}\left(2+\frac{\theta^5 x^2}{2}\right)e^{-\theta x}dx; x>0, \theta>0 \tag{1.1}$$

And it can be shown that $\int_0^\infty f(x;\theta)dx = 1$

It is a mixture of exponential and gamma distribution with mixing proportion $\frac{2}{\theta^3+2}$.

The corresponding cumulative distribution function of (1.1) can be obtained as

$$f(x;\theta) = 1 - \left[1 + \frac{\theta^4 x(\theta x + 2)}{2(\theta^3 + 2)}\right] e^{-\theta x}; x > 0, \theta > 0. \quad (1.2)$$

The graph of the probability density function and the cumulative distribution function of Exp-gamma distribution for different values of the parameter θ are shown in figure.

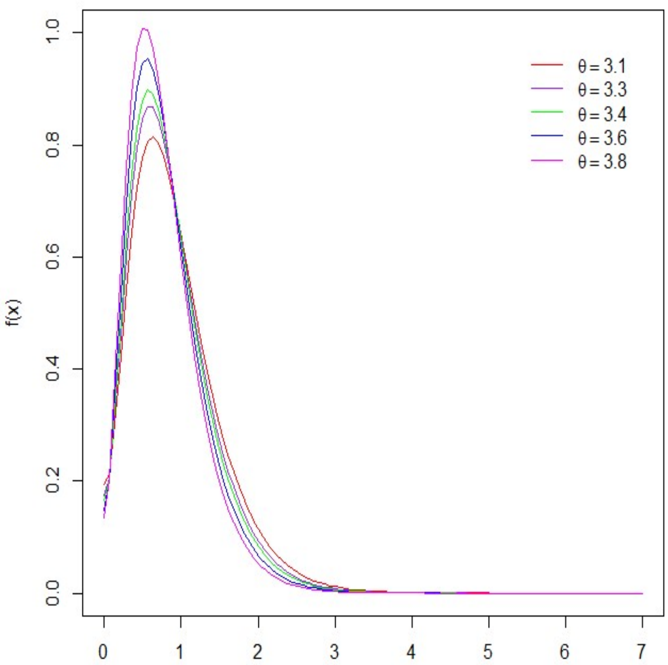

Figure 1: Graph of the pdf of Exp-gamma for $\theta > 3$.

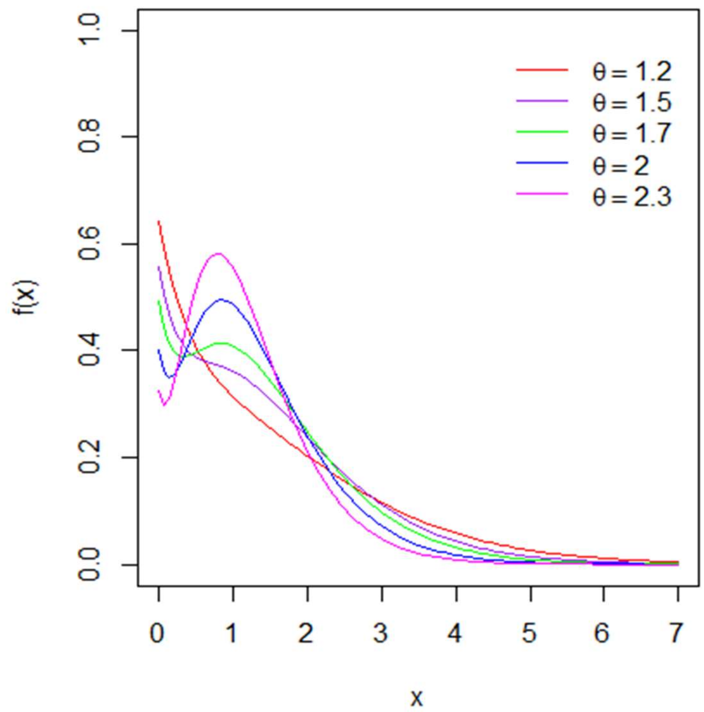

Figure 2: Graph of the pdf of Exp-gamma for $\theta > 1$ and $\theta < 3$.

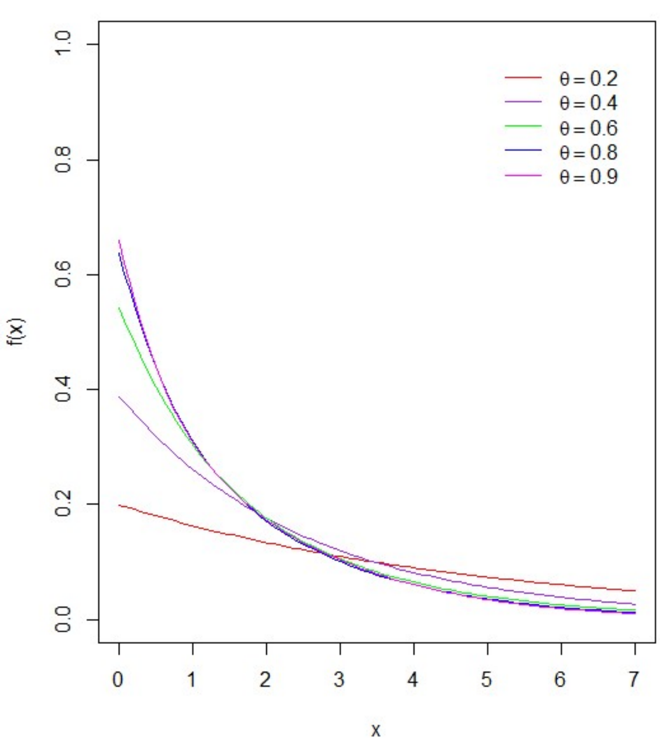

Figure 3: Graph of the pdf Exp-gamma for $\theta < 1$.

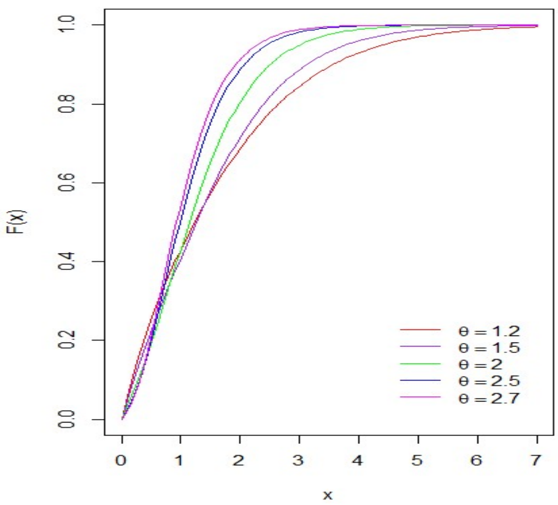

Figure 4: Graph of the cdf of Exp-gamma for the parameter θ>1.

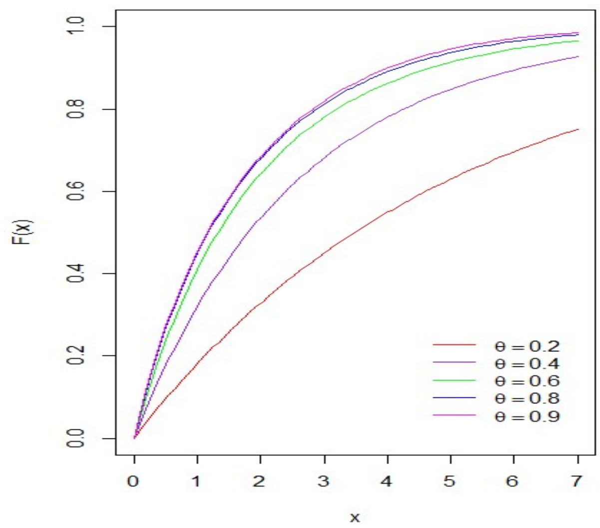

Figure 5: Graph of the cdf of Exp-gamma for the parameter θ<1.

2. Statistical Constants

The moment generating function of Exp-gamma distribution is obtained as

$$M_X(t) = E(e^{tx}) = \int_0^\infty f(x;\theta)dx$$

$$M_X(t) = \int_0^\infty e^{tx} f(x;\theta)dx$$

$$M_X(t) = \frac{\theta}{\theta^3+2}\int_0^\infty e^{-(\theta-x)}\left(2+\frac{\theta^5 x^2}{2}\right)dx$$

$$M_X(t) = \frac{\theta}{\theta^3+2}\left[\frac{2}{(\theta-t)}+\frac{\theta^5}{(\theta-t)^3}\right]$$

The r^{th} moment about origin of Exp-gamma distribution is given by

$$\mu_{r'} = \frac{4(\Gamma r+1)+(\Gamma r+3)\theta^3}{2(\theta^3+2)\theta^r} \tag{2.1}$$

Putting $r = 1$ in equation (2.1), the mean of Exp-gamma distribution is given by,

$$E(x) = \mu_{1'} = \frac{2+3\theta^3}{2(\theta^3+2)\theta^r}$$

And putting $r = 2,3,4$ the second, third and fourth raw moment are obtained as,

$$\mu_{2'} = \frac{4(1+3\theta^3)}{\theta^2(\theta^3+2)}$$

$$\mu_3{}' = \frac{12(1+5\theta^3)}{\theta^3(\theta^3+2)}$$

$$\mu_4' = \frac{24(2+15\theta^3)}{\theta^4(\theta^3+2)}$$

Using relationship between moments about mean and the moments about origin, the moments about mean of Exp-gamma distribution are obtained as,

$$\mu_2 = \mu_2' - (\mu_1^1)^2$$

$$\mu_2 = \frac{16\theta^3 + 3\theta^6 + 4}{\theta^2(\theta^3+2)^2}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

$$\mu_3 = \frac{2(18\theta^6 + 60\theta^3 + 3\theta^9)}{\theta^3(\theta^3+2)^3}$$

$$\mu_4 = \mu_4' - 4\mu_3^1\mu_1^1 + 6\mu_2^1(\mu_1^1)^2 - 3(\mu_1^1)^4$$

$$\mu_4 = \frac{3(15\theta^{12} + 144\theta^9 + 408\theta^6 + 512\theta^3 + 408)}{\theta^4(\theta^3+2)^4}$$

The coefficient of variation ($C.V$), coefficient of skewness ($\sqrt{\beta_1}$), coefficient of kurtosis (β_2) and index of dispersion (γ) of Exp-gamma distribution are obtained as

$$C.V = \frac{\sigma}{\mu_1'} = \frac{\sqrt{16\theta^3 + 3\theta^6 + 4}}{(2+3\theta^3)}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} = \frac{2(18\theta^6 + 60\theta^3 + 3\theta^9 + 8)}{\left((16\theta^3 + 3\theta^6 + 4)\right)^{\frac{3}{2}}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3(15\theta^{12} + 144\theta^9 + 408\theta^6 + 512\theta^3 + 48)}{(16\theta^3 + 3\theta^6 + 4)^2}$$

$$\gamma = \frac{\sigma^2}{\mu_1'} = \frac{16\theta^3 + 3\theta^6 + 4}{\theta(\theta^3+2)(2+3\theta^3)}$$

3. Hazard Rate Function and Mean Residual Life Function

Let X be a continuous random variable with p.d.f. $f(x)$ given by equation (1.1) and c.d.f. $F(x)$ given by equation (1.2). The hazard rate function (also known as the failure rate function) $h(x)$ and the mean residual life function, $m(x)$ of x are respectively defined as

$$h(x) = \lim_{x\to 0} \frac{P(X < x + \Delta X > x)}{\Delta x} = \frac{f(x)}{1 - F(x)} \tag{3.1}$$

And

$$m(x) = E[X = x/X > x] = \frac{1}{1 - F(x)} \int_x^\infty [1 - F(t)]dt \tag{3.2}$$

The corresponding hazard rate function, $h(x)$ and the mean residual life function, $m(x)$ of the Exp-gamma distribution (1.1) are thus obtained as

$$h(x) = \frac{\theta(\theta^5 x^2 + 4)}{2(\theta^3+2) + \theta^4 x(\theta x + 2)} \tag{3.3}$$

And

$$m(x) = \frac{2(\theta^3+2)}{2(\theta^3+2)+\theta^4 x(\theta x+2)} \int_x^\infty \frac{2(\theta^3+2)+\theta^4 t(\theta t+2)e^{-\theta t}}{2(\theta^3+2)} dt$$

$$m(x) = \frac{6\theta^3+\theta^5 x^2+4\theta^x+4}{\theta[2(\theta^3+2)+\theta^4 x(\theta x+2)]} \tag{3.4}$$

Remark:

i) $h(0) = f(0) = \frac{2\theta}{(\theta^3+2)}$.
ii) $h(x)$ is an increasing function in x for $\theta > 1$ and $\theta < 3$.
iii) $h(x)$ is an increasing function in x for $\theta > 3$.
iv) $h(x)$ is almost constant for $\theta < 1$.
v) $m(0) = \frac{2+3\theta^3}{\theta(\theta^3+2)} = \mu_1{}'$
vi) $m(x)$ is decreasing function in x for $\theta > 1$ and $\theta < 1$.

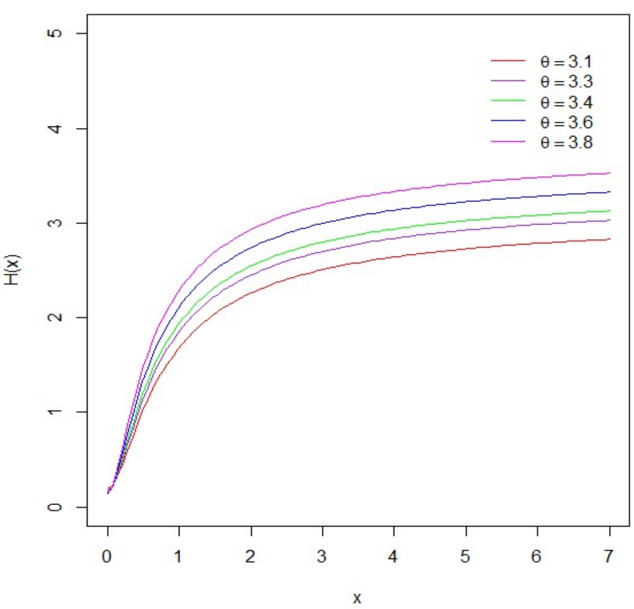

Figure 6: Graph of Hazard function of Exp-gamma for the parameter θ>3.

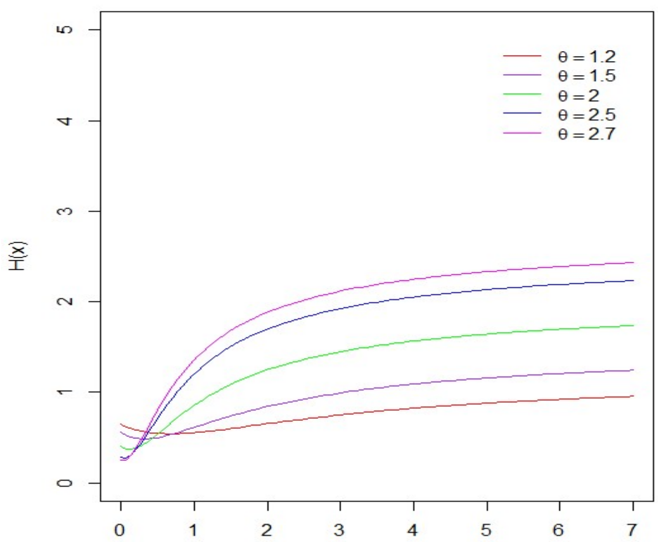

Figure 7: Graph of hazard function of Exp-gamma for the parameter θ>1 & $\theta < 3$.

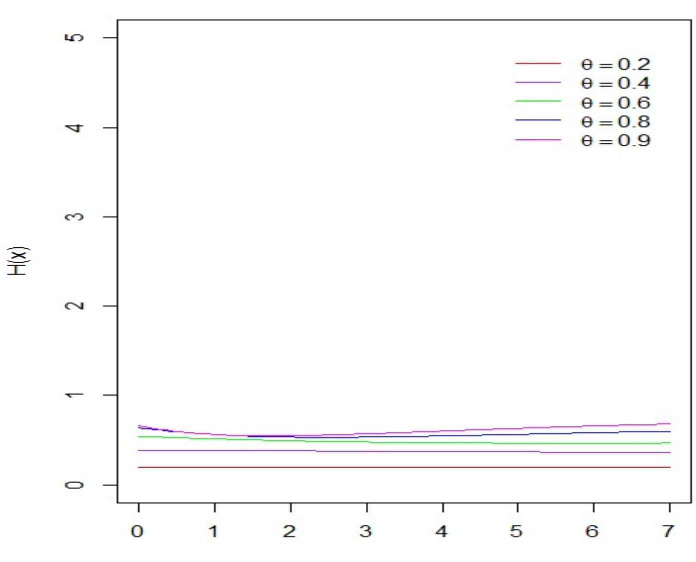

Figure 8: Graph of Hazard function of Exp-gamma for the parameter θ<1.

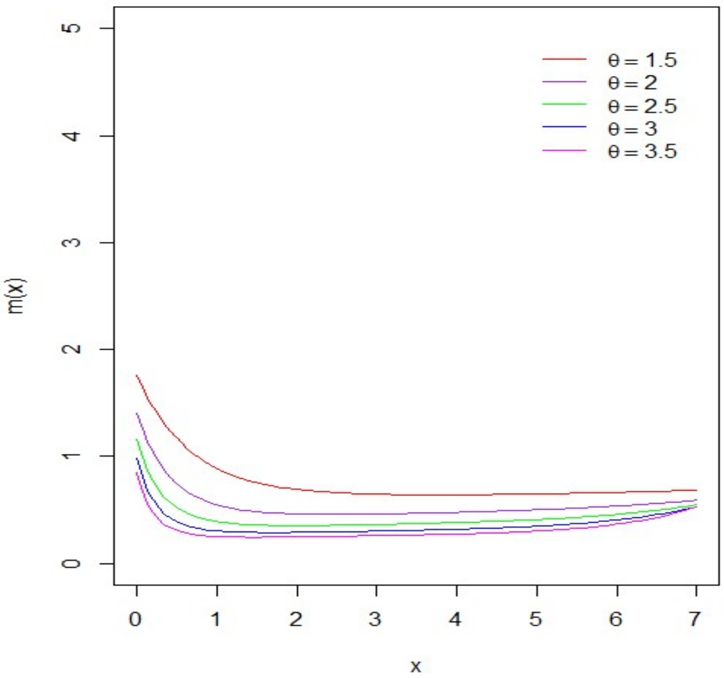

Figure 9: Graph of the MRL function Exp-gamma for the parameter θ>1.

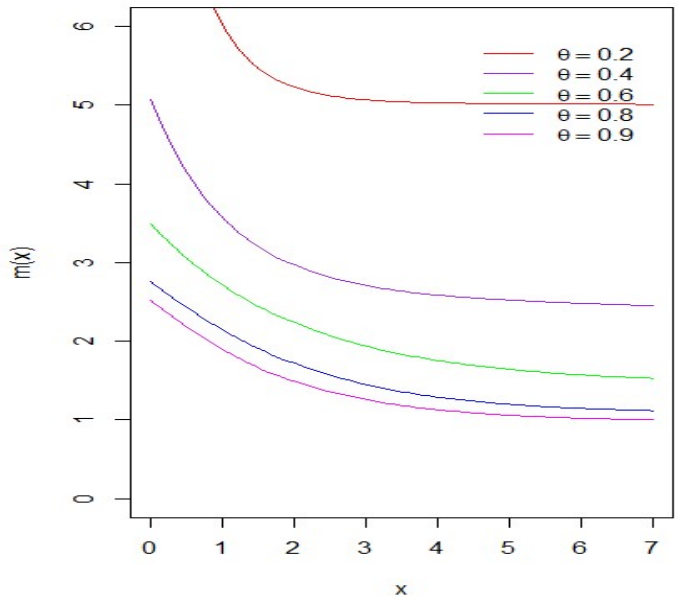

Figure 10: Graph of the MRL of Exp-gamma for the parameter θ<1.

4. Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves and Bonferroni and Gini indices have application not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance, and medicine. The equation for Bonferroni and Lorenz curve are defined as,

$$B(P) = \frac{1}{\mu}\int_0^q x\, f(x)dx = \frac{1}{p\mu}\left[\int_0^\infty xf(x)dx - \int_q^\infty xf(x)dx\right]$$

$$B(P) = \frac{1}{p\mu}\left[\mu - \int_q^\infty xf(x)dx\right] \tag{4.1}$$

And $L(P) = \frac{1}{\mu}\int_0^q xf(x)dx = \frac{1}{\mu}\left[\int_0^\infty xf(x)dx - \int_q^\infty xf(x)dx\right]$

$$L(P) = \frac{1}{\mu}\left[\mu = \int_q^\infty xf(x)dx\right] \tag{4.2}$$

Bonferroni and Gini indices are defined as,

$$B = 1 - \int_0^1 B(p)dp \tag{4.3}$$

And

$$G = 1 - 2\int_0^1 L(p)dp \tag{4.4}$$

Using p.d.f. (1.1)

$$\int_q^\infty x\,f(x)dx = \int_q^\infty x\,\frac{\theta}{\theta^3+2}\left(2+\frac{\theta^5x^2}{2}\right)dx$$

$$\int_q^\infty xf(x)dx = \left[\frac{(q^2\theta^5(\theta q+3)+(6\theta^3+4)(\theta q+1))e^{-\theta q}}{2\theta(\theta^3+2)}\right] \tag{4.5}$$

Now substituting from equation (4.5) in (4.1) and (4.2), we get

$$B(p) = \frac{1}{p}\left[1-\frac{\{q^2\theta^5(\theta q+3)+(6\theta^3+4)(\theta q+1)\}e^{-\theta q}}{2(2+3\theta^3)}\right] \tag{4.6}$$

And

$$L(p) = 1-\frac{\left(q^2\theta^5(\theta q+3)+(6\theta^3+4)(\theta q+1)\right)e^{-\theta q}}{2(2+3\theta^3)} \tag{4.7}$$

Bonferroni and Gini indices of Exp-gamma distribution are obtained as follows, by substituting from (4.6) and (4.7) in (4.3) and (4.4)

$$B = 1-\frac{\left(q^2\theta^5(\theta q+3)+(6\theta^3+4)(\theta q+1)\right)e^{-\theta q}}{2(2+3\theta^3)} \tag{4.8}$$

$$G = \frac{\{q^2\theta^5(\theta q+1)+(6\theta^3+1)(\theta q+1)\}e^{-\theta q}}{(2+3\theta^3)} - 1 \tag{4.9}$$

5. Estimation of parameter

Maximum likelihood estimate (MLE)

Let $(x_1, x_2, \ldots, x_n)$ be a random sample from Exp-gamma distribution. The likelihood function is given by,

$$L = \prod_{i=1}^{n} f(x_i/\theta)$$

$$L = \prod_{i=1}^{n}\left(\frac{\theta}{\theta^3+2}\right)\left(2+\frac{\theta^5x^2}{2}\right)e^{-\theta x}$$

$$L = \left(\frac{\theta}{\theta^3+2}\right)^n \prod_{i=1}^{n}\left(2+\frac{\theta^5x^2}{2}\right)e^{-n\theta\bar{x}}$$

The natural log likelihood function is thus obtained as

$$\ln L = n\ln\left(\frac{\theta}{\theta^3+2}\right) + \sum_{i=1}^{n}\ln\left(2+\frac{\theta^5 x^2}{2}\right) - n\theta\bar{x}$$

Now,

$$\frac{d\ \ln L}{d\theta} = \frac{2n(1-\theta^3)}{\theta(\theta^3+2)} + \sum_{i=1}^{n}\frac{5x^2\theta^4}{2\left(2+\frac{\theta^5x^2}{2}\right)} - n\bar{x}$$

Where $\bar{x}$ is the sample mean.

The maximum likelihood estimator, $\hat{\theta}$ of θ is the solution of the equation $\frac{d\ln L}{d\theta} = 0$ and it can be obtained by solving the following non- linear equation

$$\frac{2n(1-\theta^3)}{\theta(\theta^3+2)} + \sum_{i=1}^{n}\frac{5x^2\theta^4}{2\left(2+\frac{\theta^5x^2}{2}\right)} - n\bar{x} = 0$$

The exact solution of above equation for unknown parameter is not possible manually. So, above equation can be solved with the help of R Software using nlm() function.

6. Application of Exp-gamma distribution

In this section, two real life data sets are analyzed for the purpose of illustration to show the usefulness and flexibility of the Exp-gamma distribution. The newly proposed model Exp-gamma is compared with well-known one parameter distribution namely X-gamma, Exponential, Akash and Ishita. For each of this distribution parameter is estimated by using maximum likelihood method. These distributions are compared using the negative log-likelihood (-LL), Akaike information criterion (AIC) and Bayesian information criterion (BIC) which are defined by -2LL +2K and -2LL +$klog(n)$, respectively, where k is the number of parameter and n is the sample size. Further Kolmogorov –Smirnov test statistic along with associated p–value is also calculated for these datasets.

Data set 1:

The following data represents the 128 bladder cancer data by Sen et. al. (2016).

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	0.26	0.31
0.73	0.52	4.98	6.97	9.02	13.29	0.40	2.26	3.57	5.06	7.09
11.98	4.51	8.53	6.93	0.62	3.82	5.32	7.32	10.06	14.77	32.15
2.64	3.88	5.32	3.25	12.03	8.65	0.39	10.34	14.83	34.26	0.90
2.69	4.18	5.34	7.59	10.66	4.50	20.28	12.63	0.96	36.66	1.05
2.69	4.23	5.41	7.62	10.75	16.62	43.01	6.25	2.02	22.69	0.19
2.75	4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33	8.37	3.36
5.49	2.23	5.49	0.66	11.25	17.14	79.05	1.35	2.87	5.62	7.87
11.64	17.36	12.02	6.76	0.40	3.02	4.34	5.71	7.93	11.79	18.1
1.46	4.40	5.85	2.02	12.07	2.07	0.22	13.8	25.74	0.50	0.51
2.54	3.70	5.17	7.28	9.47	14.76	26.31	0.81	1.76	2.07	0.22
13.8	25.74	0.50	2.46	3.64	5.09	7.26	9.47			

Data Set 2:

The following data represents Vinyl chloride data from clean upgradient ground-water monitoring wells by Bhaumik et. al. (2009).

5.1	1.2	1.3	0.6	0.5	2.4	0.5	1.1	8.0	0.8
0.4	0.6	0.9	0.4	2.0	0.5	5.3	3.2	2.7	2.9
2.5	2.3	1.0	0.2	0.1	0.1	1.8	0.9	2.0	4.0
6.8	1.2	0.4	0.2						

Table 1: MLE of θ, -2ln L , AIC, BIC and K-S Statistic of the distribution under consideration for data set 1 and 2.

	Distribution	MLE $\hat{\theta}$	-2ln L	AIC	BIC	K-S	p-value
Data 1	Exp-gamma	0.1170046	805.8683	807.8683	810.7203	0.059897	0.748
	X-gamma	0.2860608	850.3382	852.3382	855.1902	0.18487	0.00031
	Exponential	0.1167031	805.9192	807.9192	810.7712	0.060413	0.7385
	Akash	0.3375304	903.9904	905.9904	908.8424	0.20966	0.00002594
	Ishita	0.3375304	903.9904	905.9904	908.8424	0.2297	0.000002722
Data 2	Exp-gamma	0.682126	110.6926	112.6926	114.2189	0.08882	0.9513
	X-gamma	1.0312976	112.9701	114.9701	116.4965	0.13838	0.533
	Exponential	0.5320814	110.9052	112.9052	114.4316	0.08895	0.9507
	Akash	1.165719	115.14926	117.14926	120.6756	0.15643	0.3762
	Ishita	1.165719	115.14926	117.14926	120.6756	0.13453	0.5697

It can be easily seen from the above table that Exp-gamma distribution gives better fit than X-gamma, Exponential, Akash and Ishita distribution.

7. Conclusion

In the present study, a new distribution called as Exp-gamma distribution is proposed. Some statistical properties have been discussed. Various reliability properties such has hazard rate function, mean residual life function have been obtained. The parameter of the proposed distribution is estimated by using the maximum likelihood estimation method. The proposed distribution is tested by applying it to real life data set and compared with X-gamma distribution, Exponential. Akash and Ishita distribution. It is observed that Exp-gamma can perform as good as exponential distribution and performs better than distribution like X-gamma, Akash and Ishita.

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